A NON-LINEAR STOCHASTIC ASSET MODEL FOR ACTUARIAL USE

BY S.P. WHITTEN & R.G. THOMAS

ABSTRACT

This paper reviews the stochastic asset model described in Wilkie (1995) and previous work on refining this model. The paper then considers the application of non-linear modelling to investment series, considering both ARCH techniques and threshold modelling. The paper suggests a threshold autoregressive (TAR) system as a useful progression from the Wilkie (1995) model. The authors are making available (on compact disk) a collection of spreadsheets, which they have used to simulate the stochastic asset models which are considered in this paper.

KEYWORDS

Non-linear Models; Threshold Models; ARCH Models; Wilkie Model; Time-Series Models; Price Inflation; Wages; Long-Term and Short-Term Interest Rates; Share Dividend Yields; Share Dividends.

CONTACT ADDRESSES

S.P. Whitten, M.Sc, B.Sc, Equity Research, Warburg Dillon Read, 1 Finsbury Avenue, London EC2M PPP, U.K. Tel: +44(0)171 567 8000; Fax: +44(0)171 568 4800; E-mail: Simon.Whitten@wdr.com

R.G.Thomas, B.Sc., F.I.A, Varsity Actuarial Limited, The Radfall, Summer Lane, Tyler Hill, Canterbury, Kent, CT2 9NH, U.K. Tel: +44(0)1227 463214; Fax: +44(0)1227 463214; E-mail: R.G.Thomas@ukc.ac.uk

1 INTRODUCTION

1.1 *Introduction*

1.1.1 Over the past decade the use of stochastic modelling techniques has become increasingly widespread amongst actuaries. Most notable is the pioneering work of A. D. Wilkie, who developed the first comprehensive 'Stochastic Asset Model for Actuarial Use' (Wilkie, 1986,1995).

1.1.2 Wilkie proposed a system of linear time series models based on the methods of Box and Jenkins (1978). The model has come to be widely used in actuarial work and is hence a benchmark for future development. This has prompted several investigations into its adequacy, uncovering a number of concerns leading to the conclusion of Geoghegan et al. (1992) that 'considerably more research is required in this area.' Several alternative investment models for United Kingdom data have been suggested (Smith (1996), Dyson and Exley (1995), Clarkson (1991) etc).

1.1.3 Tong (1990) in his book *Non-linear Time Series, A Dynamical System Approach* has described a class of non-linear time series models based on what he calls the 'threshold principle'. It has been increasingly recognised in scientific circles that many series previously represented linearly can be modelled better by non-linear methods. However, there is as yet little awareness of these methods amongst actuaries, although some actuarial advocates of non-linearity have developed similar ideas independently of the statistical literature (Clarkson, 1991).

1.1.4 In this paper we suggest a non-linear stochastic asset model for actuarial use. Our purposes in doing so are:

— to introduce threshold modelling to the actuarial profession, and illustrate how this can complement or replace methods based on autoregressive conditional heteroscedasticity as suggested by Engle (1982); and
— to encourage discussion and experimentation amongst actuaries on the use of non-linear models.

1.1.5 The problem of constructing a comprehensive investment model for actuarial use is formidable. As a minimum, we need a multivariate model for equities, bonds and price inflation. Ideally we would also like to include other asset classes, such as property and index-linked bonds. In addition, any application where there are liabilities dependent on wages (eg final salary pension funds) requires a model for wage inflation. Unfortunately, the data available for property and index-linked bonds comprise fewer observations than are necessary to fit non-linear models. We therefore restrict ourselves to considering the series for price inflation, wage inflation, share dividends, share yields, consols yields and base rates.

1.1.6 The rest of this paper is structured as follows. In Part 2 we outline the Wilkie model and previous work on refining the model. Some of the limitations which previous authors, including Wilkie himself, have identified are pointers to areas in which non-linear modelling might offer improvements over linear methods. In Part 3 we consider non-linear modelling, including both ARCH techniques and threshold modelling. Part 4 outlines our proposed adaptations of the Wilkie (1995) model to allow for non-linearities. Part 5 gives simulated results for our model, and some discussion of those results. Our conclusions are given in Part 6.

2 THE WILKIE MODEL AND PREVIOUS REFINEMENTS

2.1.1 Wilkie's model displays a cascade structure for the investment series, shown in Figure 2.1. Inflation is postulated as the driving force for the other series. The relationships between the variables are based on a blend of statistical evidence and economic beliefs. We will outline the structure of the Wilkie model and some of its aspects about which concern has been expressed; the reader is referred to Wilkie (1995) and Huber (1997) for a thorough investigation.

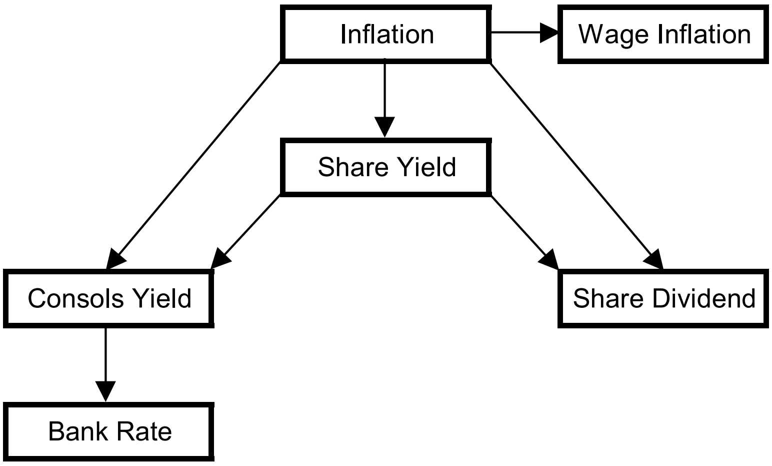

Figure 2.1. The ‘cascade’ structure of the Wilkie model.

2.2 *Price Inflation.*

2.2.1 Wilkie’s AR(1) price inflation model is of the form:

$$Q(t) \;=\; Q(t-1)exp\{I(t)\}$$

so that $I(t) = lnQ(t) - lnQ(t-1)$ is the force of inflation over the year $(t-1,t)$:

$$I(t) \;=\; QMU + QA.(I(t-1) - QMU) + QSD.QZ(t)$$

The model is economically plausible. It assumes that inflation, being a symptom of economic instability, depends only on past values of itself. There is significant autocorrelation at lag 1, which provides statistical justification for inclusion of the $I(t-1)$ variable, and no other economically plausible autocorrelation or partial autocorrelation is significant at 95%. From an economic viewpoint, autocorrelation at lag 1 is institutionalised by (for example) the use of the annual rate of inflation for the previous 12 months as the ‘headline’ rate, and the practice of many enterprises in reviewing, and often altering, their prices at annual intervals (Wilkie, 1995).

2.2.2 For the inflation model, Wilkie (1995) suggests parameters $QMU = 0.047$; $QA = 0.58$; $QSD = 0.043$. We follow Wilkie in defining ‘neutral’ initial conditions set at what their long-term means would be if all the standard deviations were zero (Wilkie, 1995, 902).

2.2.3 The inflation force data are not particularly well behaved. Geoghegan et al. describe the existence of ‘large irregular shocks’ and ‘bursts of inflation’ suggesting that a linear model cannot adequately model the data. Huber (1997) raises his concern over the stability of the parameters *QMU* and *QA*.

2.2.4 In the discussion of Wilkie (1995), Tong states in relation to linear modelling that, “...it does not matter where you are; if you are asked to make a forecast, you will give exactly the same prediction interval regardless of your current position.” The identical shapes of the conditional probability functions shown in Figure 2.2 highlight this limitation of all linear models.

2.2.5 Wilkie (1995) noted that the residuals $QZ(t)$ may not be independent, and that allowing for heteroscedasticity may be necessary. To allow for the apparent non-stationarity in the variance QSD^2, he endeavoured to fit an ARCH model of the form:

$$I(t) = QMU + QA.(I(t-1) - QMU) + QSD(t).QZ(t)$$
$$QSD(t)^2 = QSA^2 + QSB.(I(t-1) - QMU)^2$$

2.2.6 Wilkie suggests the parameters $QMU = 0.04$; $QA = 0.62$; $QSA = 0.0256$ and $QSB = 0.55$. The conditional probability functions for this model are shown in Figure 2.3.

2.2.7 ARCH models are well known to be effective in dealing with conspicuously fat tailed data. Engle (1982) in his original paper introducing ARCH models, finds evidence of conditional heteroscedasticity in UK inflation. Using the ARCH model at ¶2.2.5 above corrects most of the non-normality of the residuals (Jarque-Bera statistic is 3.2 $\Rightarrow$ $p = 0.2$), indicating that some heteroscedasticity was originally present in the inflation data. However, we find that it does not fully explain what appear to be secular changes in QMU, and asymmetries still remain in the residuals.

2.2.8 The ARCH model appears to give a better representation of inflation than the models assuming constant variance. Huber (1997) writes, "…the ARCH model appears to describe the data better than the original model. Thus it should generally be used in applications of the model, unless the ARCH effect is not significant for those particular applications."

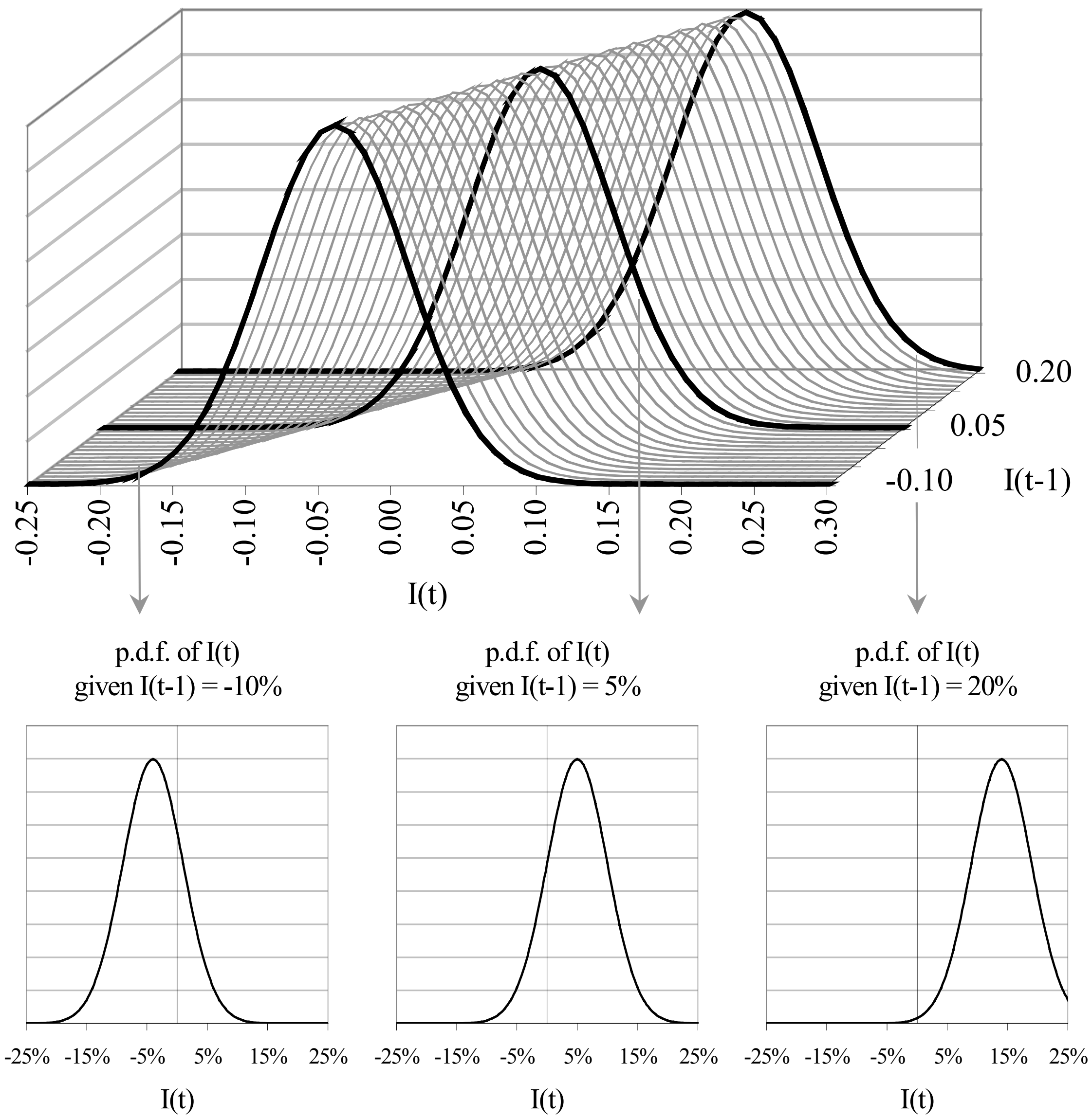

Figure 2.2. The probability density functions (p.d.f.s) of Wilkie's AR $I(t)$ model conditional on $I(t\text{-}1)$

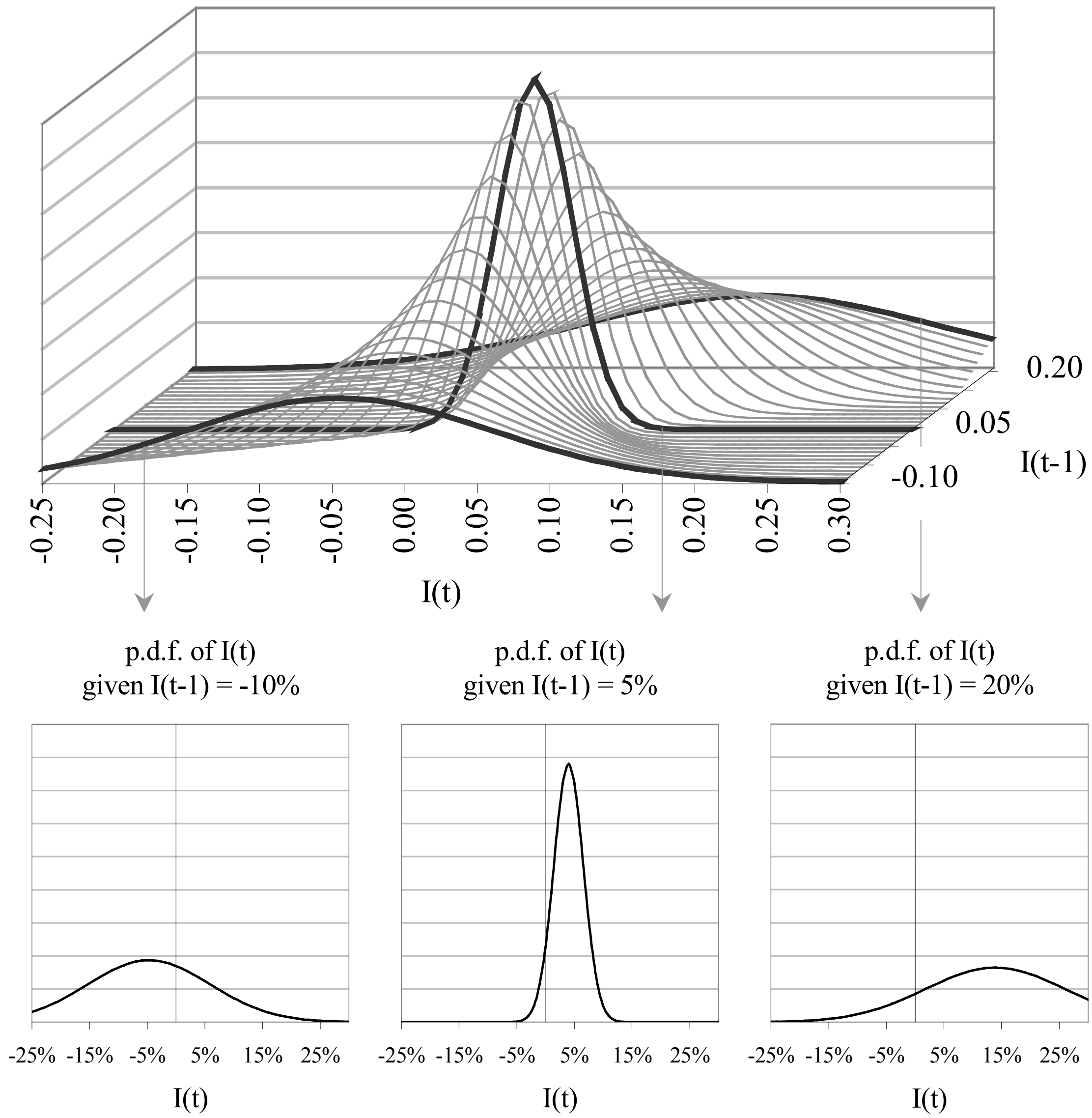

Figure 2.3. The probability density functions (p.d.f.s) of Wilkie's ARCH *I*(*t*) model conditional on *I*(*t*–1)

2.2.9 However, this kind of ARCH model can generate some unrealistic scenarios, because large shocks tend to snowball, leading to even larger shocks. Wilkie (1995, ¶11.5.2) notes that, "…on occasion, they [ARCH models] produced very large values for *QSD*, which derived from large previous values of *I*(*t*), and resulted in excessively large subsequent values, rather like a hyperinflation. This occurred twice in the 1,000 simulations."

2.2.10 In our experience, these unwanted extreme projections occur with a frequency considerably higher than 2 in 1,000. We find that 550 out of 10,000 simulations over 50 years contain at least one annual inflation projection of over 40%. The model can also generate very severe deflations: for example, if $I(t)$ = 8% and the ARCH model generates the successive shocks $QZ(t)$, $QZ(t+1)$,... = 2,2,2,2, –2 then the sequence of annual inflation figures is 8.3%, 15.3%, 30.3%, 68%, 184% and –56.3%.

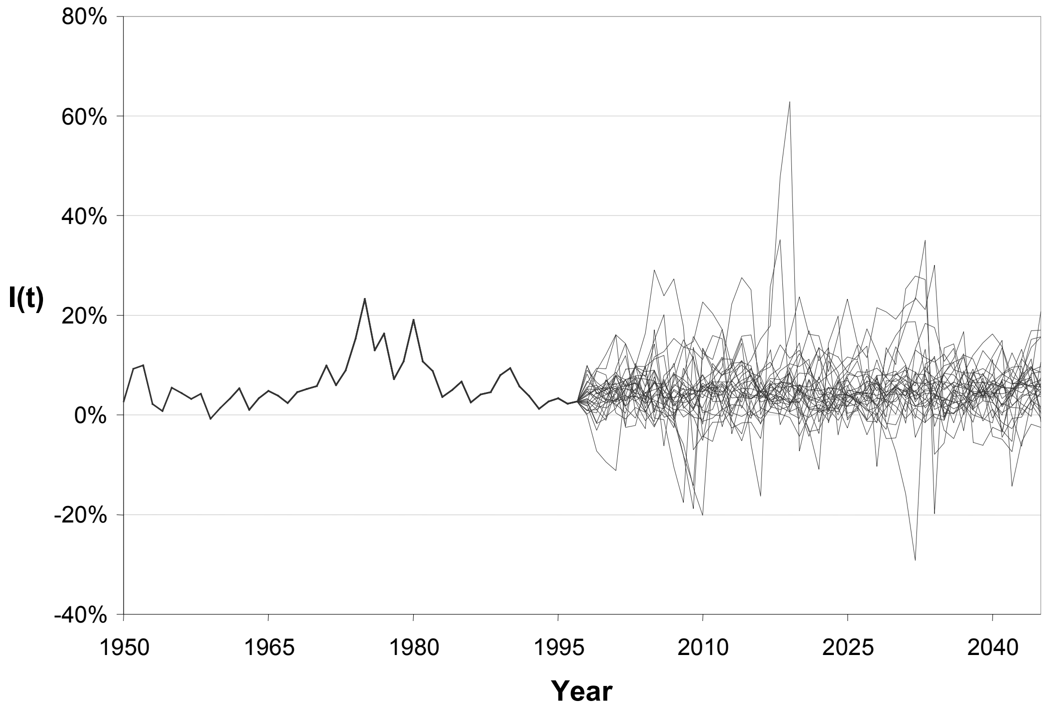

Figure 2.4. Annual force inflation 1950-1997, and 30 simulations 1997-2045 of Wilkie's ARCH model.

2.2.11 The sample of projections shown in Figure 2.4 gives another impression of the high volatility inherent in the ARCH model. Since it is the inflation process that effectively drives the whole Wilkie model, it is paramount that this model is a good representation. We believe that inflation should be modelled as a heteroscedastic process; however, ARCH models may be too volatile, and are often difficult to tame.

2.3 *Wage Inflation*

2.3.1 Wilkie's model for the wages index $W(t)$ is of the form:

$$W(t) \quad = W(t-1).exp\{J(t)\}$$

so that $J(t) = \ln W(t) - \ln W(t-1)$ is the force of wage inflation over the year $(t-1, t)$ and

$$\begin{aligned} J(t) &= WW1.I(t) + WW2.I(t-1) + WN(t) \\ WN(t) &= WMU + WA.(WN(t-1) - WMU) + WSD.WZ(t) \end{aligned}$$

2.3.2 Wilkie's preferred parameters are WMU = 0.021; WA = 0; $WW1$ = 0.6;

$WW2 = 0.27$ and $WSD = 0.233$.

2.3.3 Wilkie's wage inflation model appears to fit the data quite well. Other than some non-gaussian behaviour in the residuals, the model seems to be satisfactory.

2.4 *Share Yields*

2.4.1 Wilkie's AR(1) share dividend yield model is of the form:

$$\begin{aligned} \ln Y(t) &= YW.I(t) + YN(t) \\ YN(t) &= \ln(YMU) + YA.(YN(t-1) - \ln(YMU)) + YSD.YZ(t) \end{aligned}$$

2.4.2 Wilkie suggests the parameters $YMU = 0.0375$; $YA = 0.55$; $YW = 1.8$; $YSD = 0.155$.

2.4.3 In his 1997 review of Wilkie's model Huber notes that, "The problem with including YW is that it results in a general tendency for changes in yield to be correlated with changes in inflation, but this increase only seems to be appropriate for large increases in yields and inflation." He implies that YW is not stable, and should be different in times of high levels of inflation, as compared with in more normal times. Later in this paper we show that by a change in asset model structure, a better fit for share yields can be achieved.

2.5 *Dividends*

2.5.1 If we define $K(t)$ as the logarithm of the increase in the share dividends index from year t–1 to year t, Wilkie's MA(1) dividend yield model is of the form:

$$\begin{aligned} K(t) &= DW.DM(t) + DX.I(t) + DMU + DY.YE(t\text{-}1) + DB.DE(t-1) + DSD.DZ(t) \\ DM(t) &= DD.I(t) + (1-DD).DM(t\text{-}1) \end{aligned}$$

2.5.2 Wilkie suggests the parameters $DX = 0.42$; $DD = 0.13$; $DMU = 0.016$; $DY = -0.175$; $DB = 0.57$; $DSD = 0.07$, and $DW = (1-DX)$ to ensure unit gain.

2.5.3 We find that although $DM(t)$ can be justified in an economic sense, it is difficult to justify its inclusion on statistical grounds.

2.5.4 The dividend residuals ($DE(t)$) are fat tailed, negatively skewed and normality is rejected at 95% level (Jarque-Bera = 0.017).

2.6 *Long Term Interest Rate* (*consols yield*)

2.6.1 Wilkie's AR(1) consols yield model is of the form:

$$\begin{aligned} C(t) &= CW.CM(t) + CR(t) \\ CM(t) &= CD.I(t) + (1-CD).CM(t-1) \\ \ln CR(t) &= \ln CMU + CA.(\ln CR(t-1) - \ln CMU) + CY.YE(t) + CSD.CZ(t) \end{aligned}$$

2.6.2 Wilkie suggests the parameters $CW = 1$; $CD = 0.045$; $CMU = 0.0305$; $CA = 0.9$; $CY = 0.34$ and $CSD = 0.185$.

2.6.3 The consols yield data is not easy to model: a non-stationary mean and non-constant variance may be present. Differencing the series is not appropriate as this leads to an infinite variance structure.

2.6.4 Wilkie defines the logarithm of the real interest component $\ln CR(t)$ as a linear AR(1) or AR(3). This component is non-negative ($e^{\{\ln CR(t)\}} > 0, \forall CR(t)$). However, in prolonged periods of low or negative inflation the *nominal* interest rate in the model can become negative.

2.6.5 Wilkie (1995) shows concern that the parameter *CY* may not be significant. Huber (1997) comments that "*CY* seems to describe mainly the event that the largest increase in interest rates coincided with the largest residual from the share dividend yield model…[and that]…the relationship described by *CY* does not appear to be particularly robust." However, the popular belief that shares and bonds tend to move together provides some economic justification for the inclusion of the term.

2.6.6 The consols yield model errors ($CZ(t)$) are correlated with those from the inflation model ($QZ(t)$), suggesting that the correlation between these two variables is not fully accounted for by $CM(t)$. Wilkie (1995) writes that it may be appropriate to try different values of *CD* to correct this apparent misspecification.

2.7 *Short Term Interest Rate* (*Bank Rate*)

2.7.1 Wilkie's AR(1) cash model is of the form:

$$\begin{aligned} B(t) &= C(t).exp\{-BD(t)\} \\ BD(t) &= BMU + BA.(BD(t-1) - BMU) + BSD.BZ(t) \end{aligned}$$

2.7.2 Wilkie suggests the parameters $BMU = 0.23$; $BA = 0.74$ and $BSD = 0.18$.

2.7.3 One criticism is that the model for the bank rate relies on an adequate model for long-term interest rates. However, other than a little heteroscedasticity, on the whole the model for the bank rate seems quite satisfactory.

3 NON-LINEAR MODELLING

3.1 *Introduction*

3.1.1 The terms *linear* and *non-linear* tend to be bandied about rather loosely by actuaries, and sometimes seem to be used almost as terms of abuse. It is as well to be clear about what is meant by the descriptions linear and non-linear. Some definitions are given in Appendix A.

3.1.2 Wilkie (1995, Appendix C) describes his methods for parameter estimation, model fitting and diagnostic testing for the process of building a linear stochastic asset model. It seems generally agreed that interpretation of the statistical methods used is subjective, and final decisions should be made upon a compromise between statistical result and economic beliefs.

3.1.3 Tong (1990) proposes many different non-linear model classes. Economic theory provides only limited help in narrowing the range of plausible models. Chatfield (1995) shows concern over this model uncertainty and gives caution to those who use the process of 'data mining'. This refers to the use of fast computers to consider a large number of models, then selecting a model on the basis of statistical fit. If the use of a model produced by mining the data can be justified in an economic sense then this model becomes more valid.

3.1.4 When dealing with complicated non-linear systems of models, finding analytic solutions for each variable's mean, variance, covariance or other statistics is often very difficult and time consuming. One can get a good idea of the distributions involved by examining 10,000 random sample paths over a 50-year period. Plots of the type of Figure 2.4 also provide us with a quick idea of the shape and confidence intervals of the proposed model.

3.1.5 In describing the following non-linear models, we endeavour to adhere to the notation generally used for the Wilkie model. In each of the model definitions, we use the parameters of an inflation model.

3.2 *ARCH*

3.2.1 Autoregressive conditional heteroscedasticity (ARCH) models were originally introduced by Engle (1982), who noticed that when working with macroeconomic time series the size of the residuals seemed to be related to the size of recent residuals. They have since become a useful tool in many branches of econometrics. Conditional heteroscedasticity means that the variance of the dependent variable is not assumed constant over time (Bollerslev, Chou & Kroner, 1992). In this section we describe our experiments with ARCH models, and the problems to which they give rise.

3.2.2 There are many ways one can represent the process of conditional variance. We have considered both symmetric and asymmetric models. The most simple and commonly used model from the ARCH family is the first order generalised autoregressive conditional heteroscedastic model, or GARCH(1,1) process.

3.2.3 The conditional variance in the GARCH(1,1) process is defined by:

$$QSD(t)^2 = QSC + QSB.QE(t-1)^2 + QSA.QSD(t-1)^2.$$

Where $QE(t\text{-}1)^2$ is the squared residual the previous time period, which is known as the ARCH term; and $QSD(t\text{-}1)^2$ is the conditional variance at the previous time period, which is known as the GARCH term.

3.2.4 Wilkie uses a similar structure for his ARCH model (noted in ¶2.2.5 above). He does not use a GARCH term, and prefers $(I(t\text{–}1) - QMU)^2$ over $QE(t\text{–}1)^2$ for the ARCH term. His model is fairly easy to implement and understand. However, it can be a little erratic, producing too many hyperinflations and also a few hyperdeflations (for an example, see ¶2.2.10).

3.2.5 We can extend the ARCH family by considering processes where an asymmetric process is used to model the variance. One way of representing asymmetry is by introducing a threshold binary variable. Rabemananjara & Zakoian (1993) investigate the threshold ARCH class (TARCH(1,1)) where the form of the expected variance of the process is:

$$QSD(t)^2 = \begin{cases} QSC + QSB1.QE(t-1)^2 + QSA.QSD(t-1)^2 & QE(t-1) > 0 \\ QSC + (QSB1 + QSB2).QE(t-1)^2 + QSA.QSD(t-1)^2 & QE(t-1) \leq 0 \end{cases}$$

3.2.6 The purpose of the threshold in this model is to generate asymmetry in the errors. When the lagged error is negative, the impact of the ARCH term is ($QSB1$ + $QSB2$), but when the lagged error is zero or positive, the impact is just $QSB1$. In other words: a negative shock last period will have a different effect on the volatility in the present time period than a positive shock of equal size. There is strong rationale for the inclusion of TARCH models in many econometric contexts. For example, if one is modelling equity prices, it may be the case that downward movements of the market are followed by higher (or lower) volatilities than upward movements of the same magnitude.

3.2.7 One can include exogenous variables in the specification of conditional variance. For example, one might expect the volatility of interest rates to react to unexpected inflationary pressure the previous year.

3.2.8 A problem that is fundamental to all ARCH models is their complex structure. An iterative estimation process is necessary (we use the Berndt-Hall-Hall-Hausman (1974) routine). Also with such a small data set, the parameters are unstable. As mentioned earlier, ARCH models can also generate some unrealistic economic scenarios. We believe that these factors warrant an alternative approach to the representation of non-linearities in economic data, which we term the threshold principle.

3.3 *The 'Threshold Principle'*

3.3.1 Tong (1990) introduces the threshold principle to decompose a complex stochastic system into simpler subsystems. These models have the ability to reproduce many effects in economic models that linear AR(p) models cannot, for example asymmetries, jump phenomena and business cycles. He suggests a number of model classes based on this principle. The first class he described is known as the Self-Exciting Threshold Autoregressive Model (SETAR).

3.3.2 SETAR(n; $p_1,\ldots,p_n$) models for $I(t)$ contain n regimes separated by the $n\text{–}1$

thresholds $QR_1 < QR_2 < \ldots < QR_{n-1}$, where each regime is represented by a linear autoregressive model of order p_i ($i = 1,\ldots,n$). The active regime is identified by the threshold variable $I(t–d)$, where d is known as the delay parameter. The model is 'self exciting' in the sense that the times at which a 'jump' from one regime to the next occur are lagged values of $I(t)$ itself, rather than some exogenous variable. $I(t)$ may be described as piecewise linear.

3.3.3 One of the simplest kinds of SETAR models is the SETAR(2;1,1) model with delay, $d = 1$. This model has the form:

$$I(t) = \begin{cases} QMU1 + QA1.(I(t-1) - QMU1) + QSD1.QZ(t) & I(t-1) \leq QR \\ QMU2 + QA2.(I(t-1) - QMU2) + QSD2.QZ(t) & I(t-1) > QR \end{cases}$$

3.3.4 Here $QMU1$ and $QMU2$ are the constant means of the lower and upper regimes. $QA1$, $QA2$, $QSD1$ and $QSD2$ are the autoregressive and standard deviation parameters for the lower and upper regimes. QR is the threshold, and $I(t–1)$ is the threshold variable that dictates which regime the process is in at time t.

3.3.5 Unfortunately evaluating the threshold position(s) is not a trivial matter. Several methods have been proposed, see Tong (1990), Petruccelli & Davies (1986) and Tsay (1989). We use an adaptation of Ruey Tsay's method. Essentially the data e.g. $I(t)$, is sorted into ascending order $I(\pi)$, then $I(\pi + 1)$ is recursively regressed on $I(\pi)$. That is, starting with just a few of the data points, we repeat the regression many times, adding the next higher datum from $I(\pi)$ to the regression on each recursion. The thresholds are then shown as 'changes in direction' in the scatter plot of recursive t-ratios of AR coefficients versus $I(\pi)$. For a linear series, the t-ratios increase or decrease gradually and smoothly as the recursion continues and more data are added. If threshold behaviour is present, the t-ratios will behave exactly as those for a linear series until the recursion reaches the threshold value, QR. Once QR is reached, the estimate for the AR coefficient starts to change and the t-ratios start to deviate from their previous smooth progression. (The idea will become clearer to the reader when an example arises in Part 4 below.) This procedure and other tests for threshold behaviour are illustrated in Tsay (1989).

3.3.6 SETAR(2;1,1) models are heteroscedastic when $QSD1 \neq QSD2$. This is a very simple type of conditional variance; but it is a large improvement on constant variance, and we think it is easier to comprehend and control than using GARCH representations.

3.3.7 Another class of threshold model introduced by Tong is that of the threshold autoregressive system. For simplicity we use the version discussed by Chen (1993). This model takes the form:

$$I(t) = \begin{cases} QMU1 + QA1.(I(t-1) - QMU1) + QSD1.QZ(t) & \boldsymbol{Z(t)} \leq QR \\ QMU2 + QA2.(I(t-1) - QMU2) + QSD2.QZ(t) & \boldsymbol{Z(t)} > QR \end{cases}$$

where $Z(t)$ is a function dictating the regime of $I(t)$.

3.3.8 This class can be extended for n thresholds. Notice that when $Z(t)$ is a lag variable $I(t–1)$, this model reduces to a SETAR(2;1,1). When dealing with financial series, it is often plausible that exogenous variables may dictate the regime of threshold models. For example, one may believe each of our variables to follow a different process during a period of hyperinflation, than at a time of normal inflation. Here $Z(t)$ could be a process depending on the present and historic force of inflation.

3.3.9 After the threshold has been chosen, fitting threshold models is quite a simple process. For example, to fit a threshold model for inflation with $QR = 0.1$, we first filter the data into two regimes, conditional on whether $I(t - d) < QR$, then we use conventional linear modelling techniques (ordinary least squares) on a discontinuous sample.

3.3.10 Tong (1990) briefly suggests that it may be useful to incorporate two or more different non-linearities in the same model. If two model types are used, he calls the resulting model second-generation. He suggests a SETAR-ARCH model, where the SETAR effect concentrates on the conditional mean and the ARCH effect the conditional variance. Li and Li (1996) in their paper '*A Double Threshold Autoregressive Heteroscedastic Model*' combine SETAR and ARCH in modelling daily changes in the Hang Seng Index, and suggest a framework to follow when fitting such models. But in the present application, we find difficulty in fitting a model as exotic as this to the limited data set available to us.

3.3.11 Threshold models are straightforward (especially relative to ARCH models) to estimate, and have many desirable model qualities. However, as far as we are aware they have been little used by actuaries. Apart from simple lack of awareness, there are some more justifiable reasons for this: in most financial applications it is not immediately obvious where to fit the positions of the thresholds, or indeed what the threshold variable should be. This contrasts with some of the earlier applications, such as the modelling of rainfall, temperature and river flow in Iceland (Tong, 1990), where there is an obvious physical justification for a temperature somewhere around freezing point to be a threshold. There has also been a lack of software support for testing and estimating threshold behaviour (other than the STAR3 package offered to readers of Tong (1990) we do not know of any software for the purpose of fitting threshold autoregressive models).

3.3.12 A by-product of this research is a collection of spreadsheets (available on compact disk) which demonstrate the simulation of Wilkie's model and our threshold model (see ¶5.4.1).

4 A NON-LINEAR STOCHASTIC ASSET MODEL

4.1.1 We suggest that the economy behaves differently in times of hyperinflation, than it does in times of 'normal' inflation levels. By definition, this belief cannot be incorporated into linear models. Wilkie's linear model is widely used and for the most part a good representation of its economic variables. Having considered a variety of alternatives, we thought it best to *adapt* his model to incorporate this non-linearity, rather than fundamentally change its formulation.

4.1.2 One major concern was how to model the heteroscedasticity and apparent short term changes in the mean of $I(t)$. ARCH models are useful in representing conditional heteroscedasticity, but we find them difficult to estimate, and (as noted at ¶2.2.10 above) they can give rise to troubling results from simulation.

4.1.3 Threshold models can also represent conditional variance, and exhibit short-term changes in mean. They are an obvious choice when one wishes to represent two or more regimes.

4.1.4 We choose to model the investment series as a threshold autoregressive system. There will be two regimes proposed for each variable, conditional on whether inflation is 'normal' or 'high' at time *t*. The processes in each regime (especially the 'normal' regime) will be similar to those defined by Wilkie in his model.

4.1.5 This is not the first proposition in the actuarial literature of a threshold model for inflation. Clarkson (1991) suggests using a kind of threshold autoregressive model. His model for $I(t)$ is of the form:

$$I(t) = \begin{cases} QMU + QA.(I(t-1) - QMU) + QSD.QZ(t) + QB.Trend^{+}I(t) + QP(t).QF(t) & Trend^{+}I(t) \leq 0 \\ QMU + QA.(I(t-1) - QMU) + QSD.QZ(t) + QP(t).QF(t) & Trend^{+}I(t) > 0 \end{cases}$$

where $\text{Trend}^{+}I(t)$ (that is, in our notation, the threshold $Z(t)$), is an exponentially weighted average of historic inflation. The terms $QP(t).QF(t)$ are included in order to give inflation a 'kick' every twenty or so years. This model is an attempt at incorporating some of the non-linearities evident in the inflation data. Clarkson suggests that the inflation process follows two separate mechanisms: one when it is at normal levels, which he calls the 'quiescent phase'; and another when it is at high levels, which he calls the 'excited phase'. Clarkson's model is easy to simulate, but its complicated structure defies statistical estimation and so one cannot say which terms are significant.

4.2 *The Price Inflation Model*

4.2.1 We propose that inflation should be represented as a SETAR (self-exciting threshold autoregressive) model, with delay 1, and one threshold that differentiates between 'normal' and 'high' inflation.

4.2.2 We suggest that it seems plausible in an economic sense to have the inflation threshold somewhere between 8 and 13 percent, in the sense that inflation 'in double figures' has a psychological significance for many economic agents. To identify the most appropriate threshold, we use an adaptation of Ruey Tsay's method as described in ¶3.3.5 above. We look for turning points on the scatter plot of the t-ratios of the AR coefficients versus $I(\pi)$ to evaluate the threshold position(s). This is illustrated in Figure 4.1.

4.2.3 On examining Figure 4.1, there appears to be a change in direction somewhere between 9 and 11 percent, and perhaps around 0 percent. A threshold at 0 percent seems quite plausible in economic terms, but for simplicity we restrict ourselves to two regimes only, namely 'normal' and 'high' inflation (or in Clarkson's terminology, a 'quiescent phase' and an 'excited phase'). We suggest a threshold of 10% to partition the data. The paucity of data prevents us from being more precise. Readers may care to experiment with other values.

4.2.4 We fitted many different threshold models. Due to the paucity of data partitioned into the upper regime, it is difficult to postulate any sort of autocorrelation structure in the hyperinflation regime. The most suitable model for inflation seems to be a SETAR(2;1,0), that is:

$$I(t) = \begin{cases} QMU1 + QA1.(I(t-1) - QMU1) + QSD1.QZ(t) & I(t-1) \leq QR \\ QMU2 + QSD2.QZ(t) & I(t-1) > QR \end{cases}$$

The estimated parameters, and their standard errors are shown below:

	SETAR(2;1,0) $I(t)$	
	Parameter	Standard Error
QR	0.1	
$QMU1$	0.04	0.009
$QA1$	0.5	0.123
$QSD1$	0.0325	
$QMU2$	0.12	0.03
$QSD2$	0.05	

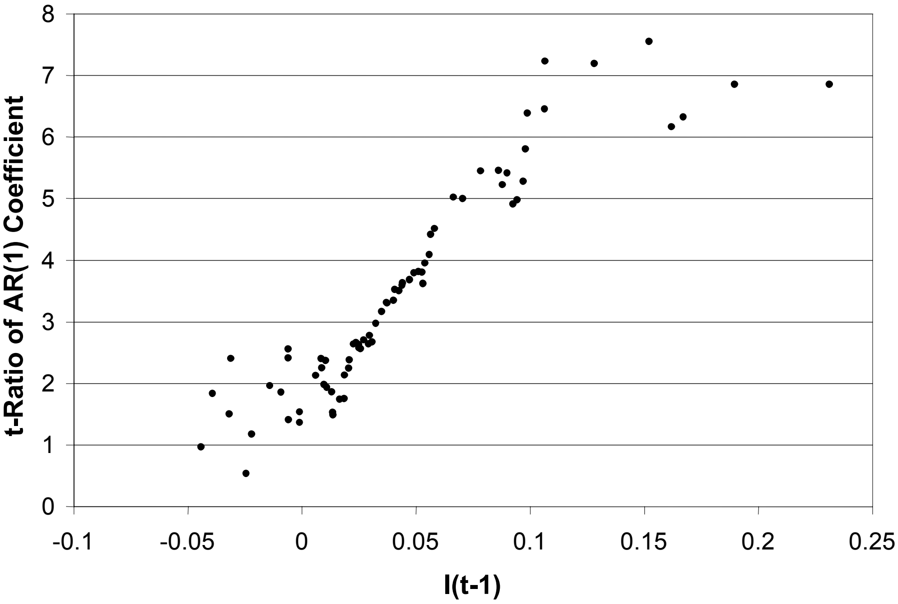

Figure 4.1. Scatterplot of Recursive t-Ratios of the Lag 1 AR Coefficient Versus Ordered $I(t–1)$ for $I(t)$, 1924-1997.

4.2.5 This model can be described in words: if inflation the previous year was below 10%, then the expected force of inflation this year is equal to its mean ‘usual’ rate (0.04), plus 50% of last years deviation from the mean usual rate, plus a random innovation which has zero mean and standard deviation 0.0325. However if inflation the previous year was above 10%, then the expected force of inflation this year is equal to its mean ‘high’ rate (0.12), plus a random innovation which has zero mean and standard deviation 0.05.

4.2.6 The residuals from the lower regime have skewness 0.08, kurtosis 2.99 giving a Jarque-Bera statistic of 0.07 implying that $p(\chi^2) = 0.96$. There is no significant residual autocorrelation, or any ARCH detected in the lower regime. The upper regime, $I(t–1) >$ 10% only holds 8 observations, which is unfortunately too few to perform any proper statistical tests. We see no economic grounds for suggesting a more complicated specification for the upper regime than the very simple one used here. The parameter values in the lower regime are quite similar to Wilkie’s usual parameters.

4.2.7 The model exhibits heteroscedasticity in a controlled way: the expected variance when inflation is in its excited phase is greater than when it is in its quiescent phase. This heteroscedastic behaviour, and the temporary change in expected mean inflation, are highlighted in the one-step-ahead probability densities shown in Figure 4.2. Notice how these density functions compare with those from the AR and ARCH models of Wilkie (Figures 2.2 and 2.3).

4.2.8 Figure 4.2 also illustrates the discrete jump in the shape of the distribution at the threshold QR. If I(t–1) is a little below QR, there is a reasonable chance that I(t) will be above QR; that is a switch to the higher regime will occur. Similarly, if I(t–1) is just above QR, there is a reasonable chance that I(t) is below QR, that is a switch to the lower regime will occur. However as k increases, the distributions for I(t+k) become more similar, leading asymptotically to a single distribution.

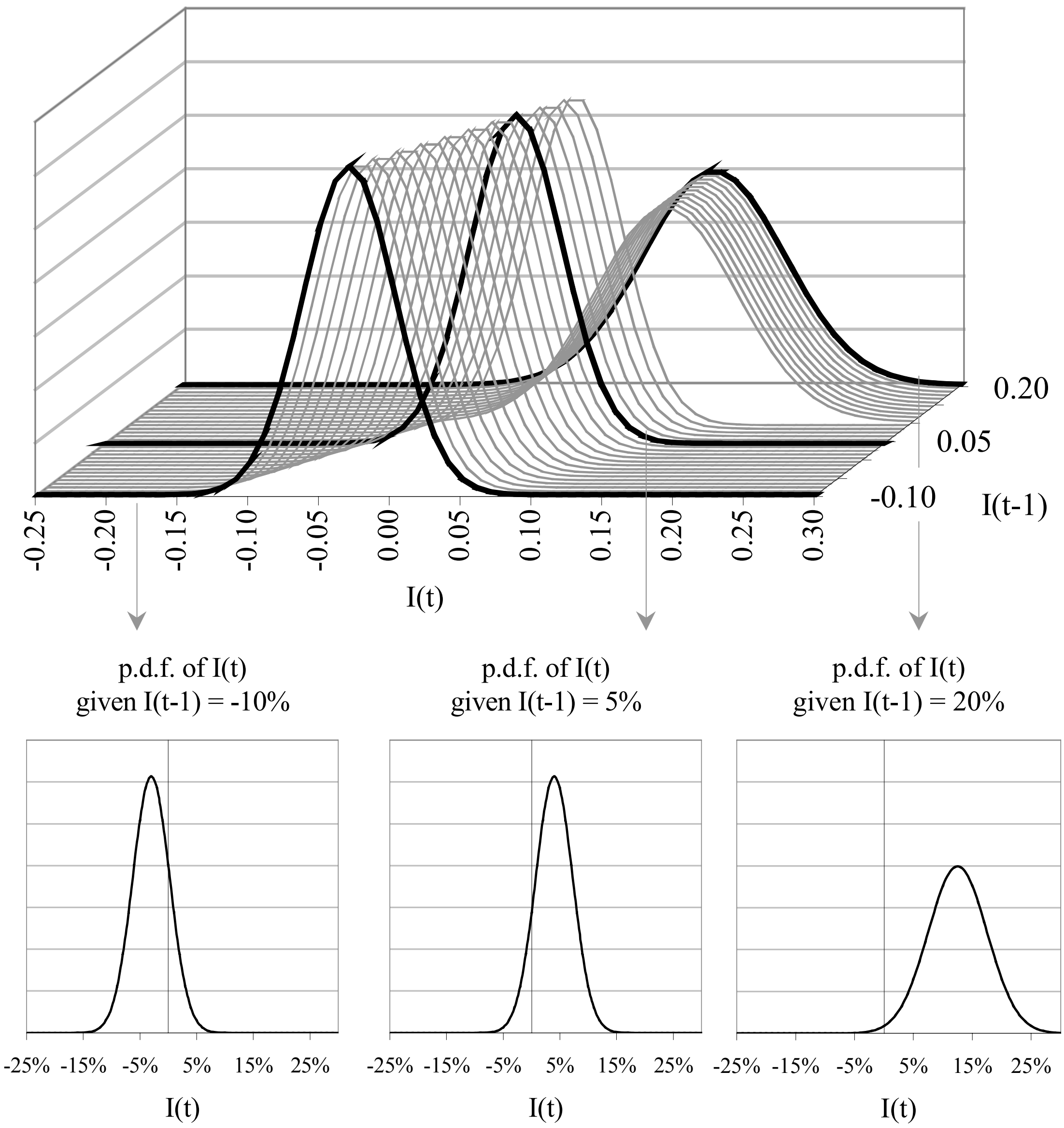

Figure 4.2. The probability density functions (p.d.f.s) of a SETAR (2:1,1) $I(t)$ model conditional on $I(t\text{-}1)$

4.2.9 When forecasting $I(t)$, we find the majority of the simulations lie in the lower regime, with the process able to switch into short excited periods of high inflation before returning to its normal quiescent phase. A visual impression of this behaviour and of how it compares with the Wilkie model can be obtained by comparing the first graph in Appendix D with the first graph in Appendix C. (Both graphs have been prepared using the same random normals.)

4.2.10 Compared with the ARCH model in ¶2.2.5, this model is less prone to generate hyperinflations and hyperdeflations such as those seen in ¶2.2.10.

4.2.11 In the remainder of Part 4, we modify Wilkie's models for $J(t)$, $C(t)$, $B(t)$, $Y(t)$ and $D(t)$ to demonstrate threshold behaviour. The models are re-estimated as a threshold autoregressive system, with threshold variable $I(t)$ and threshold QR = 10%. That is to say that each variable is expected to behave differently if inflation was high the previous year than if it were considered normal.

4.3 *The Wage Inflation Model.*

4.3.1 We did not think that it was necessary to radically change Wilkie's wage inflation model. We found that it was not necessary to have different transfer effects from inflation in each of the regimes, and have therefore kept $WW1$ and $WW2$ the same for each regime. As with Wilkie (1995) we found little autocorrelation was present after we allowed for the transfer effect from inflation. $J(t)$ was re-estimated as a threshold autoregressive (TAR) model, i.e.

$$J(t) = WW1.I(t) + WW2.I(t\text{-}1) + WN(t)$$

$$WN(t) = \begin{cases} WMU1 + WSD1.WZ(t) & I(t) \le QR \\ WMU2 + WSD2.WZ(t) & I(t) > QR \end{cases}$$

4.3.2 The estimated parameters are shown:

	TAR, $WN(t)$	
	Parameter	Standard Error
$WMU1$	0.017	0.0028
$WSD1$	0.022	
$WMU2$	0.000545	0.012
$WSD2$	0.035	

4.3.3 The residuals from the lower regime have skewness 0.3, and kurtosis 3.03 giving a Jarque-Bera statistic of 1.14 implying that $p(\chi^2) = 0.56$. There is no significant residual autocorrelation, or any ARCH detected in the lower regime. The upper regime, $I(t–1) > 10\%$ only holds 8 observations, which is unfortunately too few to perform any proper statistical tests. On economic grounds a specification of the same simple form as that in the lower regime appears reasonable.

4.3.4 However, $WMU2$ is not significant, and in practice we suggest it should be set to zero. The model can be described in words: when inflation is in its quiescent phase, the expected mean force of real wage inflation, after price inflation is accounted for is 1.7%; when inflation is considered high (above 10%) then the expected mean force of real wage inflation is zero. The model can be explained in an economic sense: lower real wage increases are acceptable to labour in times of high price inflation because of money illusion; or alternatively, this is all businesses can afford to offer when historic cost profits are squeezed by rising replacement costs of stocks of goods held. We also see that the expected standard deviation of the shock $WZ(t)$ is larger in the upper regime than in the lower regime. In economic terms, this could be interpreted as representing the difficulties which both employers and employees face in negotiating the 'right' level of wage increase in an environment of high inflation.

4.4 *The Consols Yield Model.*

4.4.1 One of our concerns about the Wilkie model is the loose connection between shares and interest rates. The only connection between the variables is the term $CY.YE(t)$ in the consols yield model. Its effect is small, and the significance of the parameter CY is questionable. Dissatisfaction with the weak transfer effect between share yields and consols yields led us to examine the correlation between them. We find a significantly strong correlation (0.53) between the log difference of the share yield and the log difference of the consols yield for the data 1923-1997, and even stronger correlation (0.63) for the post war data between 1950-1997. The positive correlation is illustrated by Figure 4.3.

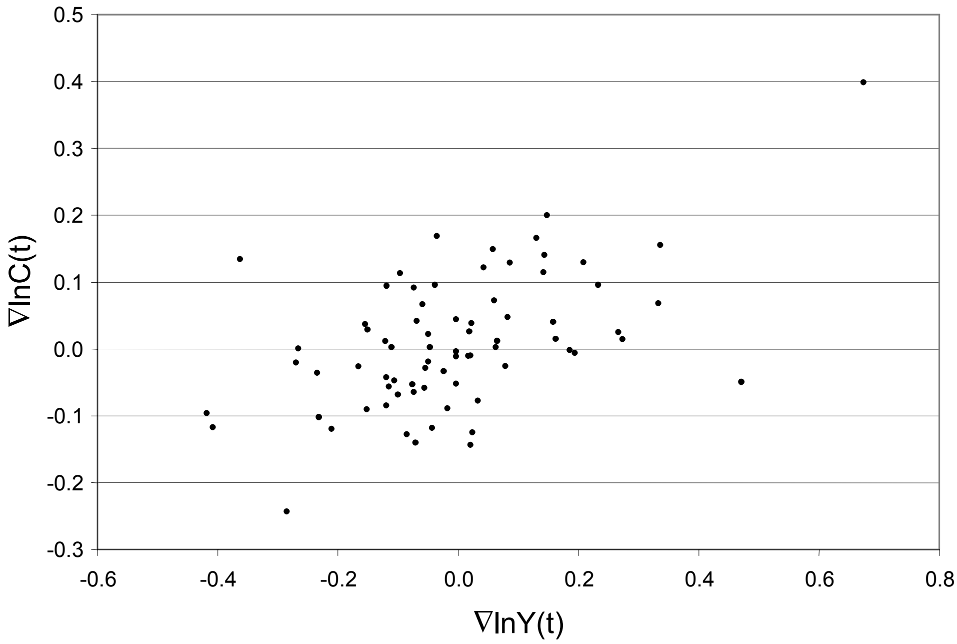

Figure 4.3. Scatterplot of the Annual Log Difference of the Consols Yield Versus the Annual Log Difference of the Share Dividend Yield, 1923-1997.

4.4.2 This is not unexpected. There is much evidence that share price changes are heavily dependent on long-term interest rates, particularly over shorter terms, whilst real factors such as earnings and dividends take longer to feed through into share prices (Pepper, 1994). We believe that share yield at time t should positively depend on interest rates at time t, and postulate a change in model structure, shown in Figure 4.4.

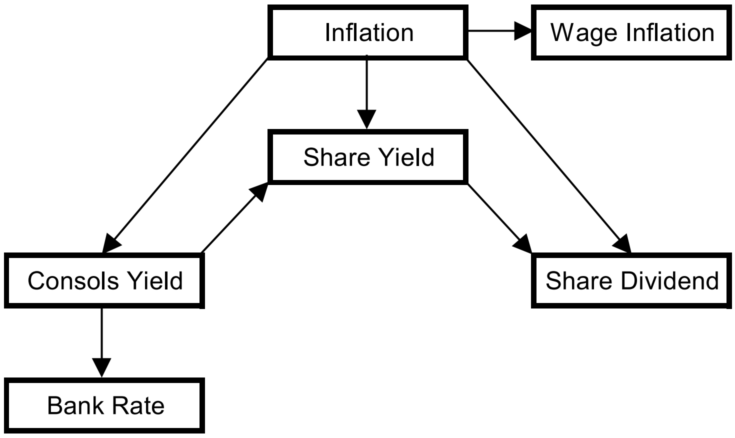

Figure 4.4. A Difference in Structure to the Wilkie Model.

4.4.3 In comparison to Figure 2.1, the only change we make is to allow the share yield to depend on the long-term interest rate, rather than vice versa. The connection between the two variables is stronger in our model than in Wilkie's ($\rho\,(\nabla \ln C(t)\,,\,\nabla \ln Y(t))$ $= 0.53 > \rho\,(\ln CR(t)\,,\,YE(t)) = 0.18$).

4.4.4 We retain the construction of Wilkie's consols yield model, insofar as we decompose the yield into two parts: an allowance for expected future inflation, $CM(t)$, and a real part, $CR(t)$. To disallow negative interest rates from occurring we have placed a reflecting barrier at zero percent, i.e. $C(t) = |CM(t) + CR(t)|$. Although Wilkie (1995) shows that there is justification to include the parameter CY, in that the log likelihood is significantly improved, we find that this is no longer the case after we implement the change in structure described in ¶4.4.2 above. The term CY is therefore excluded from the TAR model.

4.4.5 It is not easy to estimate the exponential smoothing parameter, CD, for each regime in $C(t)$. There are problems when using a more sensitive smoothing parameter, in that $\{C(t) - CM(t)\} > 0$, i.e. we cannot allow a negative real interest rate. It seemed a necessary simplification to have the allowance for expected future inflation over the life of the bond ($CM(t)$), and hence the parameter CD, defined the same for each regime. It therefore follows that, like the Wilkie model, our model gives a unit gain between inflation and interest rates.

4.4.6 $C(t)$ was then re-estimated as a TAR model, i.e.

$$\begin{aligned} \mathrm{C(t)} &= |CM(t) + CR(t)| \\ CM(t) &= CW\{CD.I(t) + (1-CD).\mathrm{CM}(t-1)\} \end{aligned}$$

$$\ln CR(t) = \begin{cases} \ln CMU1 + CA1.(\ln CR(t-1) - \ln CMU1) + CSD1.CZ(t) & I(t) \le QR \\ \ln CMU2 + CA2.(\ln CR(t-1) - \ln CMU2) + CSD2.CZ(t) & I(t) > QR \end{cases}$$

4.4.7 The estimated parameters are shown:

TAR, $\ln CR(t)$	Parameter	Standard Error
$\ln CMU1$	ln(0.027)	0.214
$CA1$	0.87	0.068
$CSD1$	0.21	
$\ln CMU2$	ln(0.061)	0.76
$CA2$	0.88	0.22
$CSD2$	0.23	

as in Wilkie (1995): $CW = 1$; $CD = 0.045$.

4.4.8 This model can be expressed in words: the absolute value of the consols yield can be decomposed into two parts: an exponentially weighted moving average component with single smoothing parameter 0.045, and a real interest component ($CR(t)$). The latter component has two regimes. If inflation is in its quiescent phase, the logarithm of the real interest component is equal to its mean (ln(0.027)) plus a proportion (87%) of last years deviation from that mean, plus 0.21 times a random standardised normal shock. However if inflation is in its excited phase, then the logarithm of the real interest component is equal to its mean ($\ln(CMU2)$) plus a proportion (88%) of last years deviation from that mean, plus 0.23 times a random standardised normal shock. The appropriate value for the parameter $CMU2$ is problematical, and we discuss this further in ¶4.4.10 below.

4.4.9 The residuals from the lower regime have skewness –0.84 and kurtosis 3.38, giving a Jarque-Bera statistic of 8.3 implying that $p(\chi^2) = 0.016$. The large negative skewness is responsible for the large Jarque-Bera statistic and signifies that there are still some problems in modelling the consols yield. As usual the upper regime, $I(t\text{-}1) > 10\%$ contains only 8 observations, and so we have to rely on economic intuition rather than statistical tests. The estimated mean real interest rate is over 6%, which suggests that in period of very high inflation, the investor expects a real return of 6%. We find this unrealistic. It occurs due to the specification of $CM(t)$, in the small smoothing value 0.045 for the parameter CD. Perhaps a larger value is required. Our TAR system is fundamentally the same as Wilkie's model, and therefore some of the limitations highlighted in Section 2.6 still remain.

4.4.10 All parameters in the consols yield model are significant at the 95% level. As with each of the other models suggested, economic judgement should be used before finalising parameter values. Due to limited data in the upper regime, the parameter $CMU2$ can accommodate a wide range of values, anything between 1% and 25% at the 95% confidence level. In our view, one should expect lower real interest rates during times of high inflation, than during times of low inflation; that is, we expect $CMU2 < CMU1$. This demands that we take a value for $CMU2$ from the lower end of the 95% confidence interval. We therefore suggest $CMU2 = 0.02$ as economically plausible; this also gives sensible confidence intervals for $C(t)$.

4.4.11 Overall, the use of a two-state model is more difficult to justify from the data for the consols yield than for the other series modelled. The *CA* and *CSD* parameters in the two states are very similar, and there is great uncertainty about the parameter *CMU2*; but if all the other series have a 'high inflation' regime, there is economic sense in the consols yield having one too. For this reason, we have retained the two-state model for inflation, but in future work we might replace the parameters in ¶4.4.7 with a single set of parameters for *CMU*, *CA* and *CSD*.

4.5 *The Bank Rate Model.*

4.5.1 We did not think that it was necessary to radically change Wilkie's bank rate model.

4.5.2 $BD(t)$ was re-estimated as a TAR model, i.e.

$$B(t) = C(t).exp\{-BD(t)\}$$

$$BD(t) = \begin{cases} BMU1 + BA1.(BD(t-1) - BMU1) + BSD1.BZ(t) & I(t) \leq QR \\ BMU2 + BA2.(BD(t-1) - BMU2) + BSD2.BZ(t) & I(t) > QR \end{cases}$$

4.5.3 The estimated parameters are shown:

	TAR, $BD(t)$	
	Parameter	Standard Error
*BMU*1	0.2	0.08
*BA*1	0.74	0.08
*BSD*1	0.17	
*BMU*2	0.15	0.31
*BA*2	0.69	0.31
*BSD*2	0.27	

4.5.4 This model can be expressed in words: when inflation is in its quiescent phase, the difference between the logarithms of long term and short term interest rates is equal to their mean difference (0.2) plus a proportion, 74%, of last year's difference from the mean difference, plus 0.17 times a standardised random normal shock. When inflation is in its excited phase, the difference between the logarithms of long term and short term interest rates is equal to their mean (high) difference (0.15) plus a proportion, 69%, of last years difference from this mean difference, plus 0.27 times a standardised random normal shock.

4.5.5 BA2 is not significantly different from BA1 and in practice we would set the parameters equal.

4.5.6 The residuals from the lower regime have skewness –0.26, kurtosis 3.65 giving a Jarque-Bera statistic of 1.9 implying that $p(\chi^2) = 0.39$. There is no significant residual autocorrelation, or any ARCH detected in the lower regime.

4.5.7 All parameters are significant at the 95% level, with the exception of $BMU2$. It is left to the discretion of the model user whether to include this parameter.

4.5.8 This model postulates a flatter yield curve in times of high inflation. It is also noticeable that the variance of the bank rate model is increased when price inflation is in the excited phase. Both of these features seem economically plausible.

4.6 *The Share Dividend Yields Model.*

4.6.1 Our share yield model is different to Wilkie's $\ln Y(t)$ in that we include a transfer effect from $\nabla \ln C(t)$ to $YN(t)$. $\ln Y(t)$ was re-estimated a TAR model, with extra parameters $YY1$ and $YY2$, to include this transfer, i.e.

$$\ln Y(t) = \begin{cases} YW1.I(t) + YN(t) \\ YW2.I(t) + YN(t) \end{cases}$$

$$YN(t) = \begin{cases} \ln YMU1 + YA1.(YN(t-1) - \ln YMU1) + YY1.\nabla \ln C(t) + YSD1.YZ(t) & I(t) \le QR \\ \ln YMU2 + YA2.(YN(t-1) - \ln YMU2) + YY2.\nabla \ln C(t) + YSD2.YZ(t) & I(t) > QR \end{cases}$$

4.6.2 The estimated parameters are shown below. We prefer to exclude $YY2$ and $YA2$, which are not significantly different from zero.

	TAR, $\ln Y(t)$ Full model		TAR, $\ln Y(t)$ $YY2$, $YA2$ excluded	
	Parameter	Standard Error	Parameter	Standard Error
$\ln(YMU1)$	ln(0.046)	0.077	ln(0.046)	*0.077*
$YW1$	–0.22	0.48	–0.22	0.48
$YY1$	0.71	0.2	0.71	0.2
$YA1$	0.71	0.085	0.71	0.085
$YSD1$	0.134		0.134	
$\ln(YMU2)$	ln(0.05)	0.2	ln(0.049)	*0.17*
$YW2$	1.9	1.5	0.65	1.3
$YY2$	–0.08	0.74	-	-
$YA2$	–0.42	0.6	-	-
$YSD2$	0.187		0.162	

4.6.3 The model (with $YY2$ and $YA2$ excluded) can be expressed in words: in any year, if inflation is in its quiescent phase, the logarithm of the share dividend yield is equal to the mean log yield (ln(0.046) and a proportion, 71%, of the deviation from this mean (after the transfer from price inflation is accounted for). To this one adds a transfer effect (71%) from the change in long term force of interest, plus 0.13 times a standardised normal shock. However, if inflation is in its excited phase, the logarithm of the share dividend yield is equal to 0.65 times the value of price inflation that year, plus the mean log yield (after the transfer from price inflation is accounted for), plus 0.16 times a standardised normal shock.

4.6.4 The residuals from the lower regime have skewness –0.37, kurtosis 3.76 giving a Jarque-Bera statistic of 3.28 implying that $p(\chi^2)=0.19$. There is no significant residual autocorrelation, nor any ARCH detected in the lower regime.

4.6.5 This model postulates that the variance of the share dividend yield is greater when inflation is high than when inflation is low. We find that the parameter representing the amount of price inflation transfer, *YW*, is not significant in the lower regime, and is best excluded. In effect, our model then substitutes a transfer effect from long-term interest rates for Wilkie's transfer from price inflation.

4.6.6 Even in the upper regime, the statistical significance of the transfer from price inflation is questionable; however, we retain it on economic grounds. We stress again that due to lack of data, the parameters in the upper regime necessarily depend as much on ones economic beliefs as on statistical estimation.

4.7 *The Share Dividends Model.*

4.7.1 As in Wilkie (1995,1986,1984), we believe that the share dividends series should be represented as an MA(1). As in Section 2.5, we define $K(t)$ as the logarithm of the increase in the share dividends index from year t–1 to t. We essentially keep the same model as Wilkie, but with a 'normal inflation' regime, and a 'high inflation' regime; on economic grounds, we impose the condition $DMU1 > DMU2$, so that dividends do better in times of normal inflation, than in times of high inflation. The standard deviation of the white noise in the lower and upper regimes were not found to be significantly different, therefore we have set the standard deviation parameter to *DSD* in both.

4.7.2 Our model for $K(t)$ is of the form:

$$K(t) = \begin{cases} DW.DM(t) + DX.I(t) + DMU1 + DY.YE(t-1) + DB.DE(t-1) + DSD.DZ(t) & I(t) \le QR \\ DW.DM(t) + DX.I(t) + DMU2 + DY.YE(t-1) + DB.DE(t-1) + DSD.DZ(t) & I(t) > QR \end{cases}$$

4.7.3 Our suggested parameters are: $DX = 0.4$; $DW = (1-DX)$; $DMU1 = 0.05$; $DMU2 = 0$; $DY = -0.2$; $DB = 0.375$; $DSD = 0.068$. $DM(t)$ is formulated as in Wilkie (1995).

4.7.4 This model can be expressed in words: in any year, the difference in logarithms between this year's and last year's share dividend is a function of current and past values of inflation. To this one adds, if price inflation is in its quiescent phase, the mean real dividend growth (0.05); or if inflation is in its excited phase, the mean real dividend growth (0). One then adds –0.2 times and 0.375 times the effects from the previous year's shares yield and share dividend errors respectively; and 0.068 times a standardised normal shock.

5 RESULTS AND DISCUSSION

5.1 *Simulated results*

5.1.1 In Table 5.1 we present descriptive statistics for each of the economic series, based on 10000 simulations of our threshold autoregressive (TAR) model. The means and standard errors are shown over various forecast horizons from 1 - 50 years.

Table 5.1. Means and standard deviations from 10000 simulations of the TAR system.

	Period 1		Period 2		Period 5	
	μ	σ	μ	σ	μ	σ
$I(t)$	0.046	0.033	0.045	0.040	0.047	0.045
$J(t)$	0.063	0.031	0.060	0.038	0.060	0.042
$C(t)$	0.077	0.005	0.077	0.007	0.077	0.011
$B(t)$	0.064	0.012	0.065	0.016	0.065	0.019
$Y(t)$	0.047	0.007	0.047	0.008	0.048	0.009
$K(t)$	0.112	0.072	0.112	0.081	0.108	0.080

	Period 10		Period 20		Period 50	
	μ	σ	μ	σ	μ	σ
$I(t)$	0.048	0.047	0.047	0.045	0.047	0.046
$J(t)$	0.061	0.044	0.060	0.042	0.061	0.043
$C(t)$	0.077	0.014	0.076	0.017	0.076	0.018
$B(t)$	0.065	0.021	0.065	0.023	0.065	0.024
$Y(t)$	0.048	0.009	0.048	0.009	0.048	0.010
$K(t)$	0.107	0.080	0.110	0.079	0.107	0.081

5.1.2 As with Wilkie's model, it is generally quite simple to adjust the parameters to reflect ones own opinions for the mean and variance of each series. With our threshold model, one has a great deal of flexibility as to what extent the non-linearity that is present in the data (mostly the 1970's period) should be included in the model.

5.1.3 The next three tables present descriptive statistics for the data series over the period 1923-1997, and for 1000 simulations of the Wilkie model and our model over 50 years.

Table 5.2 Descriptive statistics of the financial data (1923-1997).

	Data			
	Mean	S.d.	Skew.	Kurt.
$I(t)$	0.042	0.055	0.97	4.29
$J(t)$	0.057	0.051	0.65	4.05
$C(t)$	0.067	0.034	0.73	2.41
$B(t)$	0.059	0.038	0.94	2.96
$Y(t)$	0.05	0.011	0.87	4.14
$K(t)$	0.067	0.094	–1.79	8.01

Table 5.3 Descriptive statistics of the Wilkie model predictions over 50 years.

	Wilkie Model			
	Mean	S.d.	Skew.	Kurt.
$I(t)$	0.045	0.054	0.02	2.55
$J(t)$	0.061	0.053	0.05	2.62
$C(t)$	0.088	0.012	0.18	2.4
$B(t)$	0.076	0.024	0.66	3.02
$Y(t)$	0.038	0.006	0.25	2.91
$K(t)$	0.041	0.08	0.12	2.98

Table 5.4 Descriptive statistics of the TAR system predictions over 50 years.

	TAR system			
	Mean	S.d.	Skew.	Kurt.
$I(t)$	0.046	0.047	0.68	3.82
$J(t)$	0.061	0.043	0.26	3.52
$C(t)$	0.081	0.025	1.83	5.21
$B(t)$	0.065	0.028	1.15	5.32
$Y(t)$	0.048	0.009	0.76	3.90
$K(t)$	0.107	0.081	-0.05	2.99

5.1.4 The long term mean, variance, skewness and kurtosis for each series in the threshold autoregressive system are quite similar to those of the actual data. It is noticeable that both the data series (with the exception of $K(t)$) and the series in our model exhibit higher positive skewness than the series generated by the Wilkie model.

5.1.5 A special comment is in order on $K(t)$, the logarithm of the increase in the share dividends index from year t–1 to year t. The means for $K(t)$ in the data series, the TAR model and the Wilkie model stand out as much more dispersed than the other economic series. However, note that the standard errors are also large, so that the means are *not* significantly different.

5.1.6 Two sets of graphs illustrating the behaviour of the Wilkie model and the TAR model over a time horizon of 50 years are shown in Appendix C and Appendix D. By using the same sets of random normal variables for both models, these graphs give a visual impression of the differences between the Wilkie model and the TAR model.

5.2 *Refining the models*

5.2.1 The use of a fixed threshold, with substantially different models applying on either side of the threshold, inevitably gives rise to a discontinuity in the shape of our model; this is well illustrated by Figure 4.2. We acknowledge that this may be seen as unrealistic, and a useful refinement of our model would be to consider the use of a smooth transition autoregression (STAR) model, in which the switch between regimes is less abrupt (Chan & Tong, 1986). However, the paucity of data makes this difficult.

5.2.2 One can make the model continuous by putting in a logistic term, for example:

$$QMU = QMU1 + (QMU2 - QMU1)*\exp(\alpha(I(t-1) - QR)) / (1 + \exp(\alpha(I(t-1) - QR))$$

where α is a smoothing constant. If α is made small enough then the threshold model applies, but the change is continuous.

5.2.3 Our model uses the rate of inflation as the criterion for switching between quiescent and excited regimes. A more sophisticated approach to the criterion for regime switching is illustrated by Harris (1997), who suggests a Markov Chain system as the driver of the regime switches.

5.3 *Alternatives to threshold modelling*

5.3.1 We think that threshold modelling is the most readily comprehensible way of introducing actuaries to non-linear modelling, particularly if the starting point is knowledge of the Wilkie model. However, there are alternative approaches. For example, one could include quadratic or higher order powers in the model – this is called a Volterra series; or one could use state space reconstruction (Priestley, 1988). We have not done any detailed work with these alternatives, but actuaries who start with different prior knowledge to ours may find them helpful.

5.3.2 A disadvantage of threshold modelling is that leads to an increase in the number of parameters as compared with the Wilkie model. It is hardly surprising that an improved goodness of fit to past data is thus obtained, but this does not necessarily mean that the model will be any better at modelling the future. However, as we have indicated when discussing each of the individual series, we believe there is some economic motivation for postulating 'quiescent' and 'excited' regimes for inflation, and then using a different system of models for the economy in each regime.

5.3.3 An alternative perspective on the two-regime model is to see it as a pragmatic, 'least bad' way of dealing with the problem of outliers. To those who would excoriate us as data miners, we would ask: what are the alternatives?

5.3.4 There seem to be six main alternatives. First, one can use a single regime, and accept that the outliers exert a disproportionate influence on the parameter estimates (as we believe may happen in the Wilkie model). Second, one can use the Wilkie model with fatter tailed residuals. Third, one can use ARCH models; but our experience reported in Part 2 above suggests that these are no panacea. Fourth, one can use other types of non-linear model, as in ¶5.3.1 above. Fifth, one can simply discard the outliers altogether; but in any application concerned with solvency, it is the outliers which matter most. Sixth, one can follow Huber (1997, 208) in advocating that "a complete re-evaluation of economic theory and the data is required before an alternative can be suggested." We await the results of such re-evaluation with great interest; but in the meantime, we believe our approach may be the 'least bad' of the alternatives.

5.4 *Spreadsheets*

5.4.1 Others may take different views, but we hope that they will first consider investigating and experimenting with threshold models. To facilitate such experimentation, we are making available (on compact disk) a collection of spreadsheets, which we used to simulate the stochastic asset models that we consider in this paper. These user-friendly spreadsheets are menu driven and give the user a 'hands on' feel for non-linear stochastic asset modelling.

6 CONCLUSION

6.1.1 A system of threshold models can represent some of the difficult features in long-term economic series, such as asymmetry, heteroscedasticity and non-normality. Threshold features can be built around the Wilkie model in a manner which makes them easy to follow and to implement for anyone familiar with Wilkie's work. The main disadvantage is the resulting system is more complicated than Wilkie's, with an increased number of parameters.

6.2 *Further Research*

6.2.1 There are many areas in which related further research could be done. Some suggestions are listed below:

(a) investigating smooth transition autoregressive (STAR) models in place of our discrete thresholds;
(b) investigating data from other counties, especially those that have experienced high and variable inflation;
(c) fitting non-linear models to monthly UK investment data;
(d) fitting different types of threshold models to the annual UK investment data;
(e) investigating the alternative models mentioned at ¶5.3.1;
(f) investigating the non-linear models which time series analysts have proposed for other economic series (e.g. industrial production, Gross National Product (Granger & Teräsvirta, 1993)) to see what aspects of their work may be transferable to actuarial applications;
(g) application of non-linear models to higher frequency data for trading decisions.

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APPENDIX A

LINEAR AND NON-LINEAR TIME SERIES MODELS

A.1 Linear Models

A.1.1 A process (model) is linear if it is generated by the form:

$$x_t = a_0 + \sum_{j=1}^{k} a_j x_{t-j} + \sum_{j=0}^{l} b_j \varepsilon_{t-j}$$

where the a_j and b_j are constants and $\{\varepsilon_t\}$ is a series of independently and identically distributed random variables, usually with $E[\varepsilon_t] = 0$. Thus each x_t is formed as a *linear combination* of previous values of x and simultaneous and previous values of the independently and identically distributed series $\{\varepsilon_t\}$.

A.1.2 If each $\{\varepsilon_t\}$ is normally distributed, the process is said to be *Gaussian*.

A.1.3 If $E[x_t] = E[x_{t+k}]$ for all t and k, and $Cov[x_t, x_{t+k}] = Cov[x_u, x_{u+k}]$ for all t, u and k, then the process is *weakly stationary*.

A.1.4 If a process is Gaussian *and* weakly stationary, then it is also *strongly stationary*. That is, each x_t has the same distribution, and each pair x_t and x_{t+k} has a joint distribution that depends only on k.

A.1.5 However a linear model need not be stationary, and need not be Gaussian. For example, in Wilkie's model, the linear processes log *Q(t)*, log *W(t)* and log *D(t)* are not stationary. Furthermore, if the innovations *QZ(t)*, *WZ(t)* etc were picked from, for example, a t-distribution or an alpha-stable distribution (Finkelstein, 1998) the processes log *Q(t)*, log *W(t)* and log *D(t)* would not be Gaussian.

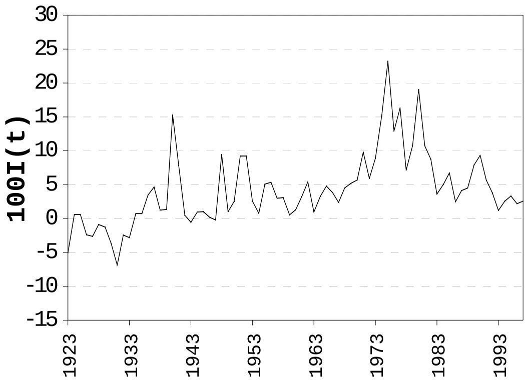
A.1.6 Wilkie's model is almost linear. However, it has many interesting and useful non-linear transformations: all the final variables, such as *Q(t), W(t), Y(t), D(t)* etc are formed by exponentiating a linear process, and so are lognormally distributed. The consols yield *C(t)* is formed as the sum of a linear process and the exponential of a linear process, and does not have a standard type of distribution.

A.2 Non-linear models

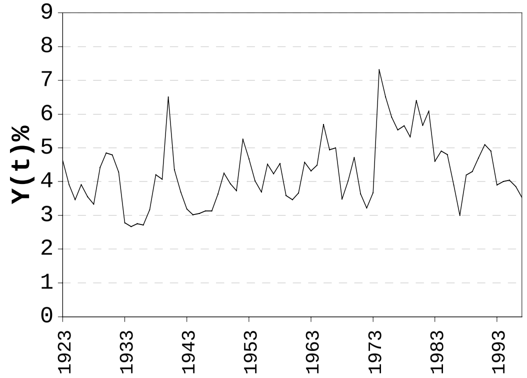
A.2.1 A model which is not linear (as defined in ¶A.1.1 above) is non-linear. There are very many different forms of non-linear model.

APPENDIX B

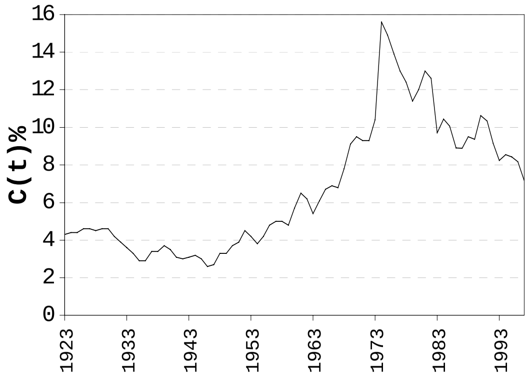
FINANCIAL TIME SERIES (1923-1997)



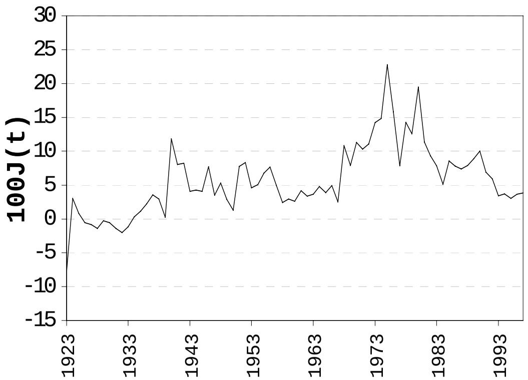
$I(t)$



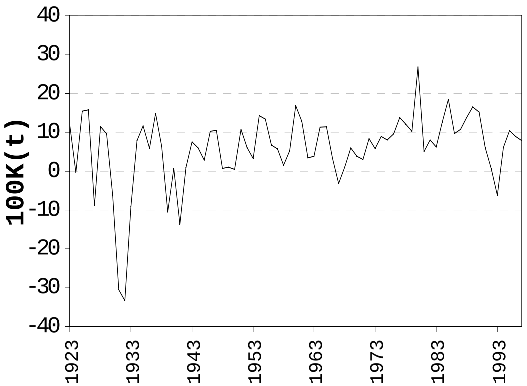
$Y(t)$



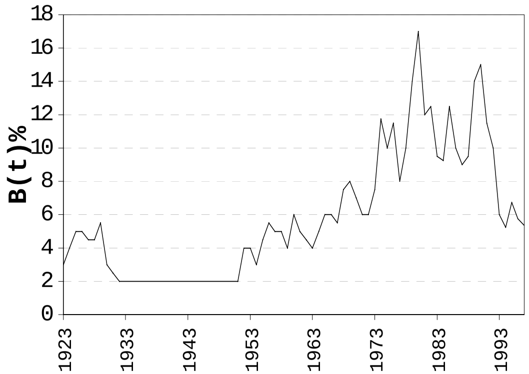
$C(t)$



$J(t)$



$K(t)$

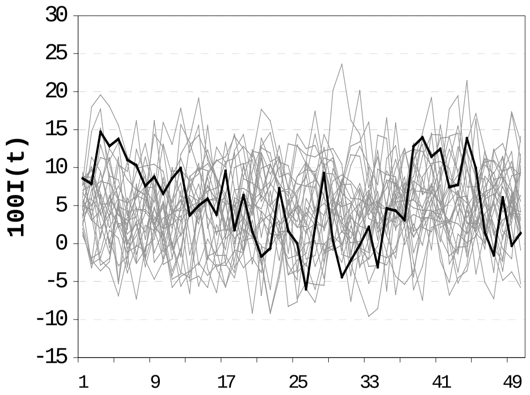


$B(t)$

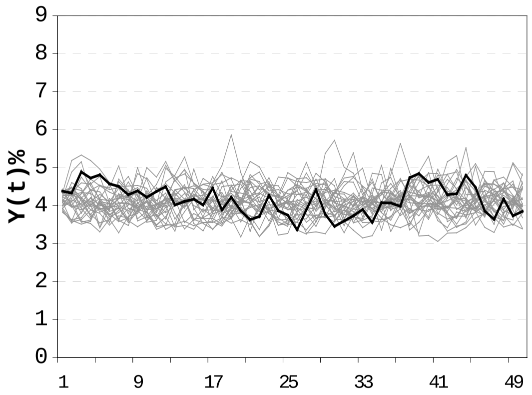
APPENDIX C

30 SIMULATIONS OF THE WILKIE MODEL

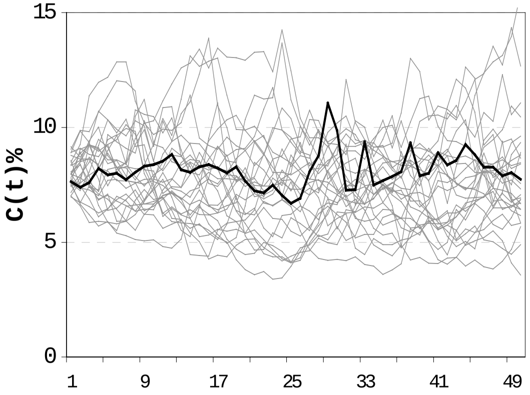
(bold lines show one realisation of the model)



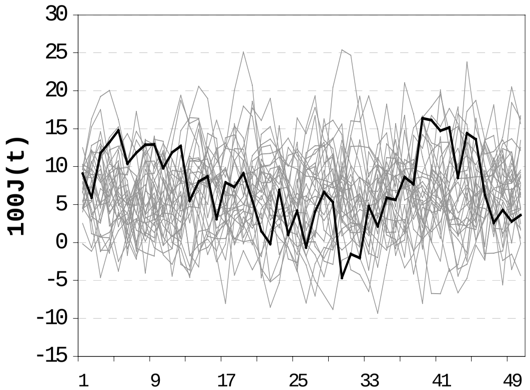
$I(t)$



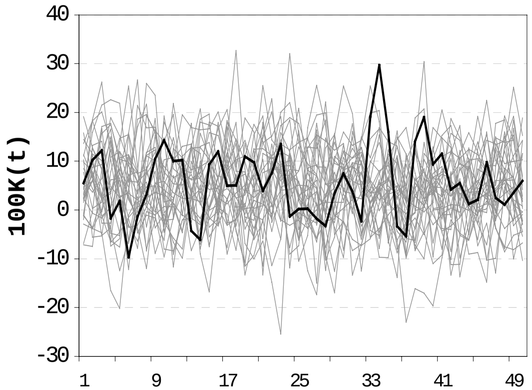
$Y(t)$



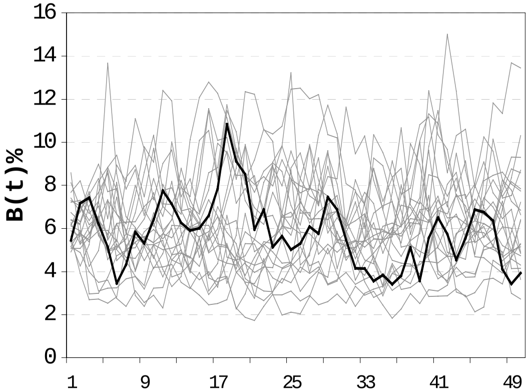
$C(t)$



$J(t)$



$K(t)$

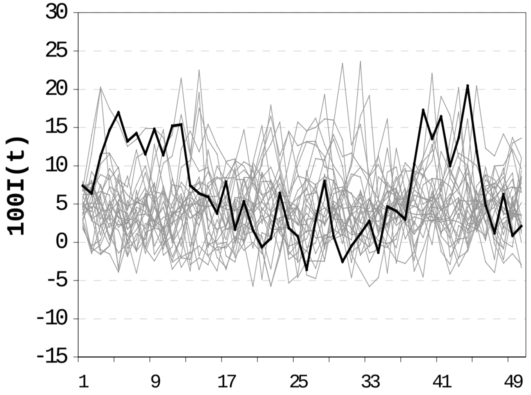


$B(t)$

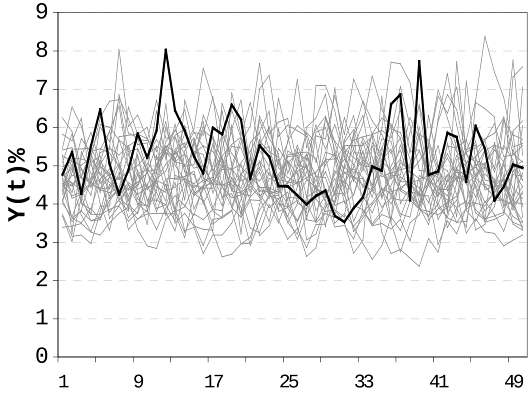
APPENDIX D

30 SIMULATIONS OF THE THRESHOLD MODEL

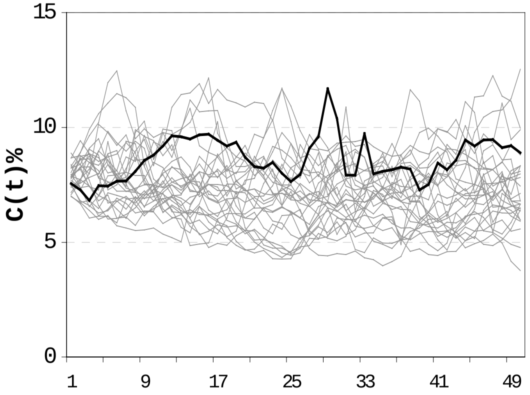
(bold lines show one realisation of the model)



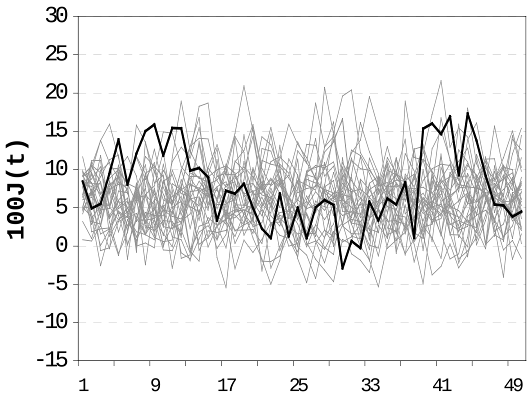
$I(t)$



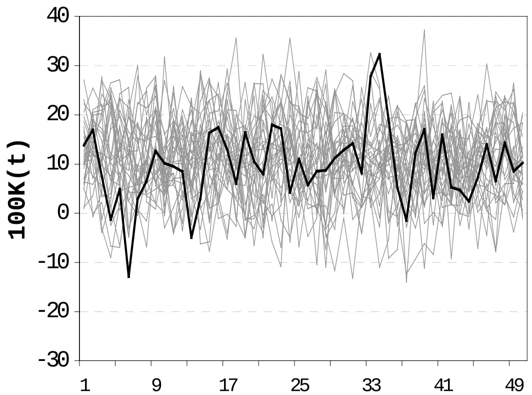
$Y(t)$



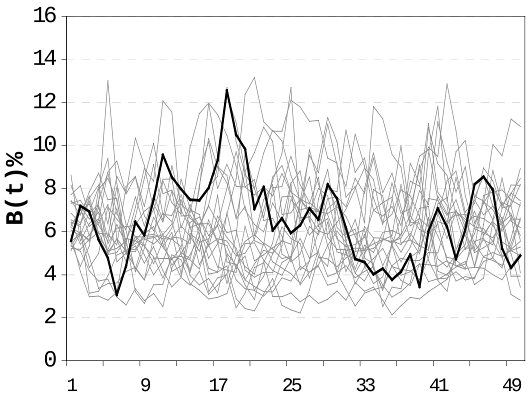
$C(t)$



$J(t)$



$K(t)$



$B(t)$