Pricing of call options with a lower reflecting barrier

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Abstract

The payoff of a call or put option can be replicated in two alternative ways: by direct replication involving a continuously varying position in the underlying asset, or by synthetic replication (defined as replication of a forward contract to buy or sell, together with replication of the complementary option). In standard models for the asset price, both ways have the same cost. But in the presence of a lower reflecting barrier under the spot price, synthetic replication is always cheaper than the direct replication for a call (and vice versa for a put). So in the presence of a lower reflecting barrier (i) a call should always be replicated synthetically (and a put directly); and (ii) put-call parity takes the form: *Synthetic Call – Put = Forward Contract*.

Keywords: Call option; Put-call parity; Reflecting barrier

1. Introduction

Thomas (2021) considered the pricing of a put option in the presence of a lower reflecting barrier under the spot price. The pricing method was to integrate the option's intrinsic value over the risk-neutralised density for the asset price at maturity. In Appendix B of that paper, I noted a puzzle: when the same method is applied to price a call option, the resulting pairs of risk-neutral call and put prices for the same strike do not satisfy put-call parity. I suggested a number of pragmatic reasons why we should nevertheless adopt the risk-neutral price of the put, and assign some residual ambiguity to the price of the call. Similarly, in an antecedent paper, Hertrich (2015, p239-240) writes that in the presence of a lower reflecting barrier "…the put price is unique and equals the risk-neutral price…the call market price can take on at least these two values, i.e., the risk-neutral call price and the imputed call price via the standard put-call parity."

The present paper suggests a resolution of the ambiguity about the price of a call, and hence the form which put-call parity takes in the presence of a lower reflecting barrier. The resolution uses the construct of synthetic replication, in contrast with the more common direct replication. For a call, direct replication involves a continuously varying portfolio which is long a fraction of the underlying asset, and short a zero-coupon risk-free bond. Synthetic replication involves two elements: entering a forward contract to buy, and purchasing a put, both with the same strike as the call we wish to replicate. The call price is then determined by replicating each of these two elements separately (via a static position for the forward, and a dynamic position for the put). An analogous synthetic replication can be constructed for a put (entering a forward contract to sell, and purchasing a call).

To state conclusions succinctly: under standard models for the asset price, both direct replication and synthetic replication have the same costs. But in the presence of a reflecting barrier, synthetic replication is always cheaper than direct replication for a call, so the price derived from synthetic replication should apply. The converse applies for a put, where direct replication is always cheaper. So in the presence of a lower reflecting barrier (i) a call should always be replicated synthetically (and a put directly); and (ii) put-call parity takes the form: *Synthetic Call – Put = Forward Contract*.

Point (ii) of the conclusions just stated may seem counter-intuitive, because the reader thinks: put-call parity is a model-free identity, so why should a lower reflecting barrier (or any other non-standard assumption for the path of the asset price) make any difference? However, we need to distinguish between:

(1) put-call parity as a *relationship between payoffs* from combinations of puts, calls and forward contracts at maturity; and
(2) the *cheapest replication schemes* for payoffs which satisfy this relationship.

The relationship between payoffs at maturity is indeed an identity, which is unaffected by the lower reflecting barrier. But the presence of the barrier changes the cheapest replication schemes for calls and puts (but not for forwards). This change drives a wedge between the pricing of options on the one hand, and forward contracts on the other hand; and this in turn implies that to satisfy put-call parity, one of the options will need to be priced as a residual item inferred from parity. This paper argues that it is always the call, not the put, that should be priced as a residual item.

The rest of this paper is organised as follows. Section 2 gives a recap of put-call parity and notes that in the absence of a reflecting barrier, risk-neutral pricing of both calls and puts will always satisfy put-call parity; it also ensures that the prices of options replicated directly, options replicated synthetically, and forward contracts are all mutually consistent. Section 3 gives formulas for direct-replication pricing of call and put options in the presence of a lower reflecting barrier, and hence shows that the mutual consistency no longer holds. Section 4 explains why. Section 5 explores an example for a call, and Section 6 briefly makes the corresponding points for a put. Section 7 states conclusions.

2. Put-call parity and risk-neutral pricing

As a reminder for the reader: put-call parity refers to the observation that if we buy a call and write a put with the same maturity T and strike price X, this is economically equivalent to entering into a forward contract which commits us to purchasing the underlying asset by paying X at time T. This is because if the asset price ends up above X, we exercise our call; if the asset price ends up below X, the put is exercised against us; so we are exposed to all the upside and downside at expiry, and we are committed to paying X at expiry. So the net cashflow for buying the call and writing the put must be equivalent to the price for entering into the forward contract. In mnemonic form, we can write put-call parity as:

$$\text{Call} - \text{Put} = \text{Forward Contract} \tag{1}$$

or symbolically:

$$C(X) - P(X) = e^{-rT}\left(F_{0,T} - X\right) \tag{2}$$

where $F_{0,T}$ is the forward price, that is the price we would agree today (but with no payment today) to pay at time T to take delivery of the asset at that time; X is the strike price we *actually* commit to paying at time T under our specific forward contract, and also the strike price of the call and put options with prices $C(X)$ and $P(X)$; and r is the risk-free rate.

Note that put-call parity does not make any assumptions about the distribution of returns on the underlying asset. As such, we expect it to apply to the prices of all options and forward contracts with the same strike, irrespective of any irregularities in the asset return (such as a reflecting barrier).

Note also that put-call parity is a constraint on pricing, rather than a complete pricing recipe: so far, we have said nothing about how we might calculate option prices which satisfy the constraint. However, under the standard assumption for the asset return (geometric Brownian motion, with no reflecting barrier), it turns out that using risk-neutral (i.e. Black-Scholes) prices for both calls and puts will always satisfy put-call parity. This was shown in Derman and Taleb (2005); the formal argument is reproduced in Appendix A.

Put-call parity implies that a call option can be replicated in two alternative ways: by direct replication involving a position in the underlying asset, or by synthetic replication (defined as a forward contract to buy, together with the purchase of a put, both with the same strike as the call we wish to replicate). The equivalence of these alternatives can be appreciated either by rearrangement of Equation (1), or by contemplating the terminal payoffs of the call and synthetic call. A similar argument applies for a put. Without a reflecting barrier, it does not matter which replication method we use for any particular option. Provided that we use risk-neutral (i.e. Black-Scholes) prices for each of the call and put, either direct or synthetic replication will give the same price.

3. A lower reflecting barrier breaks the mutual consistency of direct and synthetic replication

First, we sketch the derivation of option prices in the presence of a lower reflecting barrier. The barrier is conceived as a model of policymaker actions, such as central bank currency purchases supporting a policy floor under an exchange rate (Hertrich, 2015, Hertrich and Zimmermann, 2017), or policymaker interventions in the housing market after a large fall in prices (Thomas, 2021).

We start with a standard geometric Brownian motion for the price process. We then impose a reflecting barrier somewhere below the lower of the spot price and the strike price (i.e. $b \leq \min(S, X)$). The strike can be either lower or higher than the spot price, i.e. in or out of the money. The spot price evolves as a geometric Brownian motion, except that if the spot price hits the barrier from above, reflection occurs instantaneously, and with infinitesimal size. We can think of this as the State making a small purchase which prevents the price falling below the barrier.

The instantaneous nature of the reflection means that the price does not spend any finite time at the barrier, so no arbitrage opportunities are created (we can never buy at the barrier with certainty of a price rise). The absence of arbitrage in our model of the asset returns ensures that an equivalent martingale measure exists, in other words, a risk-neutralised version of the density for the asset price at maturity exists. So we can price options with a reflecting barrier by integrating their intrinsic values over this risk-neutral density.[1]

Thomas (2021) gives the resulting formulas for risk-neutral call and put prices. In the present paper, I state them for an asset which pays no income.[2]

For a call:

$$C_B(X) = e^{-rT} \int_X^{\infty} (S_T - X) f(S_T)\, dS_T \tag{3}$$

[1] For a fuller account of the mathematics, see Appendix A of Thomas (2021), or Hertrich (2015); or for a derivation using the partial differential equation approach, Veestraeten (2008).

[2] Ignoring income makes no difference to the fundamental points of the present paper, but simplifies some of the formulas and explanations.

$$= S\,\Phi(z_1) - X\,\mathrm{e}^{-rT}\,\Phi\left(z_1 - \sigma\sqrt{T}\right) + \frac{1}{\theta}\left\{ \begin{array}{l} +S\left(\frac{b}{S}\right)^{1+\theta}\Phi(z_2) \\ -Xe^{-rT}\left(\frac{b}{X}\right)^{1-\theta}\Phi\left(z_2 - \theta\sigma\sqrt{T}\right) \end{array} \right\} \quad (4)$$

with

X = strike price, S = current spot price, T = term, b = barrier price, r = risk-free rate, σ = volatility, $\Phi(.)$ is the standard Normal cumulative distribution function

and

$$z_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T\right] \quad (5)$$

$$z_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{b^2}{XS}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T\right] \quad (6)$$

$$\theta = 2\frac{(r-q)}{\sigma^2} \quad (7)$$

This is the Black-Scholes formula for a call, plus the $1/\theta$ term (say *Call Adjustment*), which evaluates as 0 for $b = 0$ and $b = X$, and positive everywhere in between.

Similarly, for a put:

$$P_B(X) = e^{-rT}\int_b^X (X - S_T)\, f(S_T)\, dS_T \quad (8)$$

$$\begin{aligned} &= X\,\mathrm{e}^{-rT}\,\Phi\left(-z_1 + \sigma\sqrt{T}\right) - S\,\Phi(-z_1) \\ &\quad - b\,\mathrm{e}^{-rT}\,\Phi\left(-z_3 + \sigma\sqrt{T}\right) + S\,\Phi(-z_3) \\ &\quad + \frac{1}{\theta}\left\{ \begin{array}{l} be^{-rT}\Phi\left(-z_3 + \sigma\sqrt{T}\right) \\ -S\left(\frac{b}{S}\right)^{1+\theta}\left[\Phi(z_4) - \Phi(z_2)\right] \\ -Xe^{-rT}\left(\frac{b}{X}\right)^{1-\theta}\Phi\left(z_2 - \theta\sigma\sqrt{T}\right) \end{array} \right\}. \end{aligned} \quad (9)$$

with

$$z_3 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{b}\right)+\left(r-q+\frac{\sigma^2}{2}\right)T\right] \tag{10}$$

$$z_4 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{b}{S}\right)+\left(r-q+\frac{\sigma^2}{2}\right)T\right] \tag{11}$$

This the Black-Scholes price for a bull put spread $P(X) - P(b)$, plus the $1/\theta$ term (say *Put Adjustment*). The latter term evaluates as 0 for $b = 0$ and $b = X$, and negative everywhere in between, so the put with barrier is worth a bit less than the bull put spread, as intuitively expected.[3]

Both the call and put formulas can be verified by evaluating mean option payoffs in Monte Carlo simulations of the risk-neutralised process for the observed asset price (i.e. the process in the presence of the barrier, but with the drift term set to the risk-free rate). This gives the same numerical results, subject to small error margins which reduce with increasing numbers of time steps and simulations.

The signs of the $1/\theta$ terms (positive for call, negative for put) together imply that the difference of call and put prices according to the formulas above is *larger* than for Black-Scholes. Specifically, the difference, Equation (4) less Equation (9), is

$$\begin{aligned} C_B(X) - P_B(X) = S\Phi(z_3) - Xe^{-rT} + be^{-rT}\left(1-\frac{1}{\theta}\right)\Phi\left(-z_3+\sigma\sqrt{T}\right) \\ +\frac{1}{\theta}\left\{S\left(\frac{b}{S}\right)^{1+\theta}\Phi(z_4)\right\}. \end{aligned} \tag{12}$$

This is larger than standard put call-parity $e^{-rT}\left(Se^{rT}-X\right)$, and reduces to the standard put-call parity only for $b = 0$.

[3] The intuition is as follows. Think first of the bull put spread, which involves writing a put with strike X, and protecting the downside by buying a put with a lower strike b. The payoff at maturity is at worst a loss of ($X - b$), but the price diffusion on the path to maturity can go below b (and may be below b at maturity). Then introduce the reflecting barrier $b \le \min(S, X)$. The maximum loss is the same, but the diffusion is modified so that it can never go below b. This must reduce the value of the put with barrier compared with the bull put spread, because in every scenario where the diffusion previously went below b, it is now reflected off the barrier.

So we have a puzzle: in the presence of a lower reflecting barrier, applying the standard risk-neutral pricing methodology for both calls and puts generates pairs of prices which do not satisfy put-call parity.

4. Comparing the replication of forwards and options

To resolve this puzzle, we need to think about the replication of forwards and options, first without and then with a barrier.

4.1 Forwards – static replication

In the absence of a barrier, the forward price can be established by the following no-arbitrage argument. A contract which obliges us to buy the asset by paying $F_{0,T}$ at time *T* can be hedged by going short one unit of the asset at the spot price *S* at time 0, and investing the proceeds of the short sale in a zero-coupon bond or bank account at the risk-free rate *r*. At time *T*, our short position exactly offsets the unit of the asset that the forward contract obliges us to buy, and we have earned the risk-free rate on the proceeds of the short sale. It follows that if no initial premium is required to enter the forward contract, the forward price at time 0 must be $F_{0,T} = S\,e^{rT}$. Any higher (or lower) forward price would create a static arbitrage: go short (or long) the forward and go long (or short) the spot, then hold to maturity to earn a risk-free profit.

Now suppose there is a reflecting barrier. What difference does this make? It makes no difference. Whilst any incidence of reflection at the barrier affects the spot price, it also affects the forward price in the same way. The spot-forward no-arbitrage relationship as described in the previous paragraph just described still works in exactly the same way.

4.2 Options – dynamic replication

In the absence of a barrier, the central insight of Black-Scholes is that a portfolio long one call option *C* and short fraction $\delta_{CALL} = \partial C / \partial S$ of the asset is riskless, and so must earn the risk-free return. It follows that the call option can be replicated by a replicating portfolio, comprising a fractional long position of size $\delta_{CALL} S$ in the asset, and a short position of $C - \delta_{CALL} S$ in a risk-free bond. For example, if the call option expires in the money, the hedging scheme ensures that we end up long exactly 1 unit of the asset (offsetting the 1 unit of the asset which we need to supply when the option is exercised), and short the bond in an amount equal to the strike price (offsetting the cash we receive when the option is exercised).

Now suppose there is a reflecting barrier. What difference does this make? Recall that the purpose of the short position in the hedge portfolio $\left(C-\delta_{CALL}S\right)$ is to render that portfolio riskless, so that it earns the risk-free return. Then note two points. First, what happens at the barrier is already riskless. There is no possibility of spot moving down, and also no possibility of an 'ordinary' move up; there is only the certainty of an infinitesimal and instantaneous positive intervention. Second, if the hedge portfolio were to receive any benefit from this intervention, the portfolio's instantaneous return would deviate from the risk-free rate. Both these points carry the same implication: *delta must tend to zero as spot tends to the barrier*.

Now consider the implications for the replicating portfolio. If delta tends to zero at the barrier, the replicating portfolio never receives any benefit from the interventions. So the replicating portfolio does not capture the interventions as they occur. Instead, the bond part of the replicating portfolio (and hence the pricing of the option) will need to *anticipate* the interventions.

This may initially seem surprising, because we tend to think that when replicating an option on a spot price which receives interventions, it would be beneficial for the replicating portfolio also to receive interventions. But the surprise is alleviated when we recall that the rationale of risk-neutral pricing is that the hedge portfolio at all times earns the risk-free return. The only way it can do this at the barrier is for the delta to tend to zero when spot approaches the barrier. [4]

For another perspective on this, note that the interventions represent a form of 'hybrid return', neither fully risk-free nor fully stochastic. Their *incidence* at barrier is guaranteed, but not their *quantum* over any particular term (which differs for each realised path for the asset). Hence option pricing needs to anticipate the interventions, analogous to it anticipating that the asset will earn the risk-free return.

In summary, replication of forward contracts (via static positions) ignores the interventions; risk-replication of options (via dynamic positions) must anticipate them. This inconsistency in treatment of the interventions means that the usual consistency between pricing of forwards and risk-neutral pricing of both call and put options will break down.

To understand this breakdown more thoroughly, it will help to consider the pricing of an example call option for a range of barrier levels, via the two alternative methods: direct replication of the call, and separate replication of each of the two components of the synthetic call.

[4] Appendix B of Thomas (2021) suggested that the hedge portfolio did not capture the interventions because we cannot adjust the portfolio at the precise instant when spot touches the barrier. Whilst the latter point is true in a practical sense, it is not the real reason why the hedge portfolio does not capture the interventions. The real reason is that the delta must, by design, tend to zero as spot approaches the barrier, in order for the hedge portfolio to earn at all times the risk-free rate.

5. Example call option

As a convenient example (inspired by the NNEG context, but simplified by having no income on the asset), let spot price S =1, strike X = 1 (i.e. at-the-money), barrier b = 0.5, risk-free rate r = 0.015, term T = 25 years and volatility σ = 13%.

5.1 Direct replication

The formula for a call with barrier in Equation (4) evaluates as 0.408. This can be interpreted as the price of the replicating portfolio which when dynamically adjusted over the term of the option, will replicate its payoff at maturity. The required delta is found by differentiating Equation (4) with respect to the price of the underlying asset:

$$\frac{\partial C_B}{\partial S} = \Phi(z_1) - \left(\frac{b}{S}\right)^{1+\theta} \Phi(z_2). \tag{13}$$

Note that this expression tends to 0 as $S \to b$, confirming our intuition (in Section 4.2 above) that the delta must tend to zero as spot approaches the barrier. The same is true of the delta for a put, derived in Appendix C of Thomas (2021):

$$\frac{\partial P_B}{\partial S} = \Phi(z_1) - \Phi(z_3) + \left(\frac{b}{S}\right)^{1+\theta} \left[\Phi(z_4) - \Phi(z_2)\right]. \tag{14}$$

The call delta evaluates as 0.8005 for our example. So the replicating portfolio holds 0.8005 units of the underlying asset, and has a position in the zero-coupon risk-free bond given by Equation (4) less Equation (13), or 0.408 – 0.8005 = – 0.393. The negative sign signifies a short position in the bond, i.e. in anticipation of receiving the strike at maturity, the holding in the underlying asset can be partly financed by borrowing (making the replicating portfolio and hence the call price cheaper). Monte Carlo simulations confirm that this replication scheme works as expected.

Table 1 extends this example. For a range of values of the barrier level b, it gives the fractional position in the underlying asset, the corresponding bond position, and the resulting option price. The spot price of the asset is assumed as S = 1. I make the following observations on Table 1.

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Table 1: Direct replication for an at-the-money call option, for various barrier levels

	Direct-replication portfolio for call		Price of call
Barrier level *b*	(A) Fractional long positon in underlying asset	(B) Zero-coupon bond position (–ve short, +ve long)	(A) + (B) = Equation (4)
0	0.8164	– 0.4120	0.404
....			
0.5	0.8005	– 0.3927	0.408
....			
0.79	0.5201	– 0.0250	0.495
0.80	0.5014	+ 0.0005	0.502
....			
0.9	0.2803	+ 0.3076	0.588
1	0	+ 0.7093	0.709

– The fractional position in the asset in column A for a zero barrier is just the Black-Scholes delta of 0.8164. Similarly, the bond short position of – 0.4120 in column B corresponds to that prescribed by Black-Scholes.

– As the barrier rises, the fractional position in the asset in column A falls (the figures are evaluated as δ_{CALL} as given in Equation (13)). Intuitively, we need the delta to go to zero when the barrier equals the strike, so it needs to fall continuously as the barrier gets closer.

– As the barrier rises, something needs to be set aside to anticipate the interventions (which are not captured by the fractional position in the asset). So the magnitude of the short position in the bond in column B reduces. This happens faster than the fractional position in the asset falls, and hence the price of the option in the final column increases.

– For $b > 0.79$, the bond position changes sign (i.e. it becomes a long position in the bond). This is because the amount set aside to anticipate the interventions now exceeds the amount that can be borrowed in anticipation of receiving the strike.

– In the limiting case of $b = S$ in the last row of the table, the call is equivalent to a forward. The the fractional position in the asset goes to zero. The initial replicating portfolio consists solely of an investment in the risk-free bond. (But as soon as the spot price moves up, we sell some of the bond and buy a little of the asset.)

5.2 Synthetic replication

For the simplest case of synthetic replication, consider the last row of the table, where $S = X = b = 1$. This limiting case is not realistic for NNEG valuation (or indeed any other purpose), because $S = b$ implies that the spot price can never fall below today's level. But as a thought experiment, $S = X = b$ implies a call which is certain to be exercised, and hence is equivalent to a forward contract. This suggests that rather than the price shown in the table, a suitable price for this particular call is simply the premium to enter a forward:

$$\text{Premium to enter a forward} = e^{-rT}\left(Se^{rT} - X\right) \tag{15}$$

which evaluates as 0.313 for $S = X = 1$. This compares with 0.709 for direct replication in Table 1, so synthetic replication is considerably cheaper. Any price above 0.313 will be arbitraged away. So the price of 0.313 will prevail: the call will be priced as if it were a forward contract.

Notwithstanding our preference for the price of 0.313, the dynamic replication strategy which initially invests nothing in the asset and 0.709 in a zero-coupon bond is a valid replication strategy. Under this strategy, when the asset moves up off the barrier, its delta as given in Equation (13) rises slightly from 0, and so we buy a little of the asset, financed by selling part of our zero-coupon bond. If we then adjust the position throughout the term, simulations show that our final positions in asset and bond replicate the call payoffs at maturity (subject to small hedging errors, which reduces as we increase the frequency of hedging).

Alternatively, replication using the Black-Scholes delta is another valid dynamic strategy. In this case, we begin with a higher initial delta, and a short position in the bond; again, simulations show that under the Black-Scholes replication scheme, our final positions in the asset and bond always replicate the call payoffs at maturity, subject to small hedging errors which reduce as the time step is reduced.[5]

[5] It may seem surprising that Black-Scholes reliably replicates the call payoffs at maturity in the presence of the barrier. This can be understood by noting that away from the barrier, the stochastic process for a reflected geometric Brownian motion is exactly the same as an ordinary geometric Brownian motion (GBM). Then note that Black-Scholes replication has the property that it works for *any* GBM path, including a freak path which always turns upwards at a particular level (without a barrier at that level). The performance of Black-Scholes in the presence of the barrier can then be understood as being the same as its performance for the freak path.

Both dynamic strategies – replication with our call formula, or Black-Scholes replication – are *valid* in the presence of the reflecting barrier; but they are also *unnecessarily expensive*, and therefore redundant. Rather than a dynamic strategy with a delta which goes to zero at the barrier (missing all the interventions), or a Black-Scholes strategy with a fractional delta at the barrier (partially benefiting from the interventions), we can instead note that a call with $b = S$ is equivalent to a forward contract, and so replicate its payoffs by a static long position (receiving full benefit from the interventions).

Now consider a barrier slightly below 1, say $b = 0.9$ (the penultimate row of Table 1). Direct replication gives a price of 0.588 (in the right column of the table). Alternatively, the call with $b = 0.9$ can be replicated synthetically. Suppose we enter into a forward contract which gives us an obligation to purchase the underlying asset by paying 1 at time T. This exposes us to all the upside and downside risk of the asset price at time T. So to replicate a call, we then need to remove downside risk, by purchasing a put which gives us the right (but not the obligation) to sell for 1 at time T. That is, the synthetic call is:

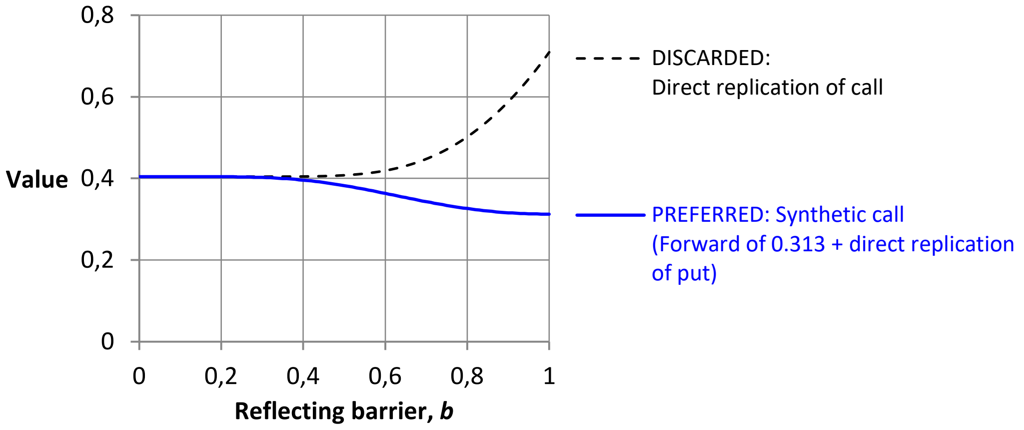
$$\text{Synthetic call} = \text{forward contract} + \text{put} \qquad (16)$$

For our example of $b = 0.9$, a put evaluated using the formula in Equation (9) is 0.003. So the synthetic call in Equation (16) evaluates as 0.313 + 0.003 = 0.316. This is less than the direct-replication price of 0.588 in Table 1. So again, the direct-replication price should be arbitraged away, and the synthetic replication price will prevail.

It turns out that for a call, the synthetic replication price remains cheaper than the direct replication price for all barrier levels down to zero (when the two converge at the Black-Scholes price). Hence a call should always be priced by direct replication. This is illustrated for our example ($S = X = 1$) in Figure 1, with all feasible barrier levels $b \leq \min(S, X)$ on the x-axis.

In Figure 1, the synthetic replication of the call appears to become cheaper as the barrier level increases (i.e. the blue line slopes downwards). This may seem counter-intuitive: won't raising the barrier raise the diffusion of the spot price, and so raise the price of a call? However, for every level of the barrier, our numeraire in Figure 1 is the current spot price *given the barrier level in force.* The synthetic call *as a fraction of the spot price* is 0.404 for $b = 0$, and 0.313 for $b = 1$; but the spot price in the latter case (in a numeraire of £) will be *more* than 0.404/0.313 = 1.29x the spot price when $b = 0$, and so the call price (in a numeraire of £) will also be higher.

Figure 1: Call option: synthetic replication is cheaper than direct replication, for all feasible levels of the reflecting barrier (example at-the-money call, $S = X = 1$)



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Another perspective on the validity of the downward-sloping synthetic replication curve comes from the framing that the synthetic replication comprises a forward contract plus "insurance" against the spot price being below the strike at maturity. Intuitively, as the barrier level is raised, the price of the "insurance" as a fraction of the prevailing spot price will fall.

5.3 Further insight: unsuitability of direct replication call price if volatility is high

Another way of seeing that the price derived from direct replication of a call in Equation (4) must be rejected is to note that for high values of the volatility parameter σ, the formula increases without limit. In particular, in our example in Section 5.1, the call formula exceeds 1 for $\sigma > 0.349$. But if the price of a call exceeds the price of the stock, this creates an arbitrage: we can write the call and buy the stock for an immediate credit ($C - S$), and then also receive either the strike (if the call is exercised) or nothing (if the call is not exercised) at maturity. The direct replication call formula is therefore unsatisfactory for high volatility.

The direct replication put formula (and hence the synthetic call) does not have this problem. As $\sigma \to \infty$, the put formula in Equation (9) increases modestly towards a limit, and remains less than the

Black-Scholes price for all feasible barrier levels ($0 < b \leq \min(S, X)$). Therefore the synthetic call (forward + put) remains less than the spot price of 1.

5.4 Further insight: synthetic replication better exploits the interventions

Another way of describing why synthetic replication is always cheaper than direct replication for a call is to say that synthetic replication better exploits the interventions at the barrier. To see this, note that direct replication portfolio holds fraction δ_{CALL} of the asset (and this fraction tends to zero as spot approaches the barrier). The direct replication portfolio therefore receives no benefit from the interventions. In contrast, the synthetic replication portfolio holds a forward contract (hedged by holding a constant 1 unit of the asset), while also holding fraction δ_{PUT} of the asset (and this fraction tends to zero as spot approaches the barrier). So the synthetic replication portfolio receives a full 1 unit benefit from all the interventions. This makes the synthetic replication scheme cheaper.

6. Example put option

The converse of the whole argument in section 5 applies for a put. Again, it is easiest first to consider the limiting case $S = X = b = 1$, that is an at-the-money put with barrier equal to the strike. Clearly, $X = b$ means the spot can never go below the strike, and so this put must be worth zero.

Now consider a lower barrier, say $b = 0.9$. The put now has a positive value, which can be directly replicated using the put delta in Equation (14). For our example, this gives a price of 0.003.

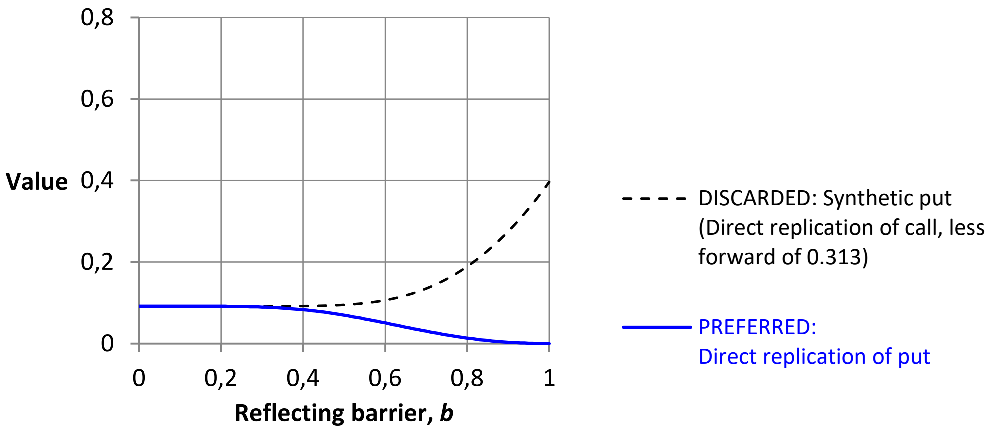
Alternatively, the put can be replicated synthetically. Suppose we enter into a forward contract which gives us an obligation to sell the underlying asset for 1 at time T. This exposes us to all the upside and downside of the asset price at time T, with a gain from downside and a loss from upside. So to replicate a put, we then need to remove the upside risk, by also purchasing a call which gives us the right (but not the obligation) to buy for 1 at time T. That is, the synthetic put is defined as

$$\text{Synthetic put} = \text{call} - \text{forward contract} \tag{17}$$

which evaluates using Equations (4) and Equation (15) as 0.588 – 0.313 = 0.275 for our example. Note that this is much dearer than the direct replication price of 0.003.

It turns out that synthetic replication remains dearer than direct replication for a put, for all barrier levels down to zero (when the two converge at the Black-Scholes price). This is illustrated for our example put (at-the-money, $S = X = 1$) in Figure 2, with all feasible barrier levels $b \leq \min(S, X)$ on the x-axis.

Figure 2: Put option: direct replication is cheaper than synthetic replication, for all feasible levels of the reflecting barrier (example at-the-money put, $S = X = 1$)



For further affirmation of sections 5 and 6, consider the results from the put and call formulas for the two extremes for the barrier (i) $b = 0$ and (ii) $b = X$. The put formula in Equation (9) gives intuitively satisfactory prices of (i) Black-Scholes and (ii) zero respectively. But the call formula in Equation (4) gives (i) Black-Scholes and (ii) a price *higher* than the price of a forward contract. The latter call price seems clearly unsuitable, because with a lower reflecting barrier equal to the strike, a call becomes economically equivalent to a forward contract. In contrast, synthetic replication of a call gives intuitively satisfactory prices of (i) Black-Scholes and (ii) the forward price.

7. Conclusions

Putting sections 3 to 6 together, we can state the following conclusions.

(1) A call option in the presence of a lower reflecting barrier should always be priced by synthetic replication, because this is cheaper than direct replication. The converse applies for a put.

(2) Equivalently, the call formula in Equation (4) should not be used (because synthetic replication is always cheaper). Conversely, the put formula in Equation (6) should be used (because synthetic replication is always dearer).

(3) Put-call parity in the presence of a lower reflecting barrier takes the form

$$\text{Synthetic Call} - \text{Put} = \text{Forward Contract} \tag{18}$$

These conclusions are analogous to those developed in Protter (2013) for option prices in the presence of bubbles in the price of the underlying asset. As with the reflecting barrier, a bubble may disrupt the standard put-call parity. Protter distinguishes between "fundamental" and "market" prices for options, where the fundamental price is the discounted expectation under the risk-neutral measure (i.e. our Equations (4) and (9) for call and put respectively). For a put, market and fundamental prices are always the same; but for a call, market and fundamental prices in the presence of a bubble differ by an amount equivalent to the difference *Synthetic Call – Call* in the present paper.[6]

Apart from their intrinsic interest, these conclusions provide additional comfort that the risk-neutral pricing formula for a no-negative-equity guarantee (i.e. a put option) with a reflecting barrier in Thomas (2021) should be robust to the possible future development of traded markets for long-term options on houses prices. In that paper, the adoption of the direct replication price was predicated partly on the pragmatic argument that forward contracts and options on house prices are not currently traded (and so put-call parity was merely an interesting theoretical point, rather than a pressing empirical concern). But if they start to trade, the present paper gives a rationale for expecting puts to be priced by direct replication, and calls by synthetic replication.

Finally, if the assumption of a reflecting barrier and the conclusions of this paper are accepted, this has an interesting corollary for the "Principle II" promulgated by the Prudential Regulation Authority regarding valuation of equity release mortgages. This is covered in Appendix B.

Computer code

Code in the R programming language implementing Monte Carlo evaluation of call and put options and demonstrating the associated replication schemes is available on the author's website.

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[6] Our Equation (16) less Equation (4), with Equations (9) and (15) substituted into Equation (16), yields Call – Synthetic Call = $S - e^{-rT}\mathrm{E}\left[S_T\right]$, where the expectation is taken under the risk-neutral measure. This corresponds to the expression for (Fundamental Price – Market Price) of a call in Equation (80) of Protter (2013)..

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Appendix A

Why risk-neutral pricing and put-call parity are mutually consistent (in the absence of a reflecting barrier)

Given the standard geometric Brownian motion model for the asset price, the mutual consistency of risk-neutral option pricing and put-call parity can be demonstrated as follows (Derman and Taleb, 2005).

Put-call parity in Equation (2) is

$$C(X) - P(X) = e^{-rT}\left(F_{0,T} - X\right) \tag{19}$$

where $F_{0,T}$ is the forward price, that is the price we would contract today (but with no payment today) to pay at time *T* to take delivery of the asset at that time*;* *X* is the strike price we *actually* commit to paying at time *T* under our specific forward contract, and also the strike price of the call and put options with prices *C*(*X*) and *P*(*X*); and *r* is the risk-free rate.

By the standard no-arbitrage argument (elaborated in Section 4.1 of the paper), the forward price $F_{0,T}$ evaluates as Se^{rT}. So we can write Equation (19) as

$$C(X) - P(X) = e^{-rT}\left(Se^{rT} - X\right) \tag{20}$$

The Black-Scholes call price is

$$C(X) = e^{-rT}\,\mathrm{E}\left[S_T - X\right]_+ = e^{-rT}\left[Se^{rT}\Phi\left(z_1\right) - X\Phi\left(z_1 - \sigma\sqrt{T}\right)\right] \tag{21}$$

where S_T is the asset price at time *T*, σ is the volatility of the asset, $\Phi(.)$ is the standard Normal cumulative distribution function, and $z_1 = 1/\sigma\sqrt{T}\left[\ln\left(S/X\right) + \left(r + \sigma^2/2\right)T\right]$.

Similarly, the Black-Scholes put price is

$$P(X) = e^{-rT}\,\mathrm{E}\left[X - S_T\right]_+ = e^{-rT}\left[X\Phi\left(-z_1 + \sigma\sqrt{T}\right) - Se^{rT}\Phi\left(-z_1\right)\right]. \tag{22}$$

When we difference (21) and (22), the sums of Normal functions (e.g. $\Phi\left(z_1\right) + \Phi\left(-z_1\right)$) each reduce to 1, giving

$$C(X)-P(X) \quad = e^{-rT}\left[Se^{rT}-X\right] \tag{23}$$

which is the put-call parity in Equation (20) above.

Now note that if we tried to evaluate the expectations in Equations (21) and (22) by hypothesising an expected growth rate μ for the asset, and a risk discount rate R, with either or both different from the risk-free rate r, then the equality in Equation (23) would no longer hold for all positive values of the strike price X. So the risk-neutral rate is the *only* assumption for asset growth rate and risk discount rate which will satisfy put-call parity for all possible strike prices. In this sense, risk-neutral pricing and put-call parity are mutually consistent.

Appendix B

PRA Principle II for no-negative-equity guarantee valuation

For certain regulatory purposes, the Prudential Regulation Authority (PRA) has promulgated four principles (labelled I to IV) for valuation of an equity release mortgage (ERM) and the no-negative-equity guarantee (NNEG) embedded within it. These principles have a common-sense appeal beyond their immediate regulatory context. However, if the assumption of a lower reflecting barrier is accepted, Principle II may be inappropriate, at least in the form usually stated.

Principle II states[7]:

> "(II) The economic value of ERM cash flows cannot be greater than either the value of an equivalent loan without an NNEG or the present value of deferred possession of the property providing collateral.
> …[This principle] is derived from the following considerations:
>
> (i) Given the choice between an ERM and an equivalent loan without an NNEG, a market participant would choose the latter, since either the guarantee is not exercised, in which case the ERM and the loan have the same payoff, or it is, in which case the ERM pays less.
> (ii) Similarly, a market participant would prefer future possession of the property on exit to an ERM, given that the property will be of greater value than the ERM if the guarantee is not exercised, or the same value if it is."

The verbal presentation of the principle obscures the underlying rationale of the dual limits as an application of put-call parity, which can be seen as follows. Define:

[7] PRA (2020), Paragraph 3.15.

X_T Rolled-up loan at maturity time T (the strike price of the NNEG)

H_T House price observed at maturity time T

$F_{0,T}$ Forward house price (price agreed at time 0, to be paid at time T, to take delivery at time T)

Then the potential payoff of an ERM in period T (the T-period 'ERM-let') is

$$\min\left(X_T, H_T\right) = X_T - \max\left(X_T - H_T, 0\right)$$

and the present value of this is

$$\text{Present value of } T\text{-period 'ERM-let'} = \mathrm{e}^{-rT} X_T - put. \tag{24}$$

In the <u>absence</u> of the reflecting barrier, this can alternatively be written as follows (by applying the standard put-call parity to re-state the *put*):

$$\text{Present value of T-period 'ERM-let'}$$
$$= \mathrm{e}^{-rT} X_T - \left[call - e^{-rT}\left(F_{0,T} - X_T\right)\right] \tag{25}$$

$$= e^{-rT} F_{0,T} - call. \tag{26}$$

By inspection of Equations (24) and (26), and noting that *put* and *call* are both always positive, we can see that these equations give two upper bounds for the present value of ERM: (i) $\mathrm{e}^{-rT} X_T$ (the present value of the loan without the NNEG) and (ii) $e^{-rT} F_{0,T}$ (the present value of deferred possession of the property). This establishes legs (i) and (ii) of Principle II.

But in the presence of a reflecting barrier, by the arguments in the body of this paper, a call option needs to be priced synthetically. If we substitute the synthetic price $(e^{-rT} F_{0,T} + put)$ for *call* in Equation (25), then Equation (26) becomes just $e^{-rT} X_T - put$, the same as the first limit, the one derived from Equation (24). So in the presence of a reflecting barrier, we can dispense with the second leg of Principle II (i.e. the prepaid forward price as an upper limit on the value of the ERM-let).

The second leg of Principle II is more likely to be relevant when the term T is large, so that the limit given $e^{-rT}F_{0,T}$ becomes small.[8] This prepaid forward price (called "deferment price" in the NNEG context) is unchanged by the barrier, and so still becomes small for long terms. But this does not matter to the ERM writer, because the presence of the barrier drives a wedge between the pricing of forwards (including prepaid forwards) and options, and in particular, puts become cheaper to hedge than in the absence of a barrier. By dynamically hedging the put it has written, the ERM writer can be sure of receiving $e^{-rT}X_T - put$ (where the *put* valuation allows for the barrier); the ERM writer does not need to be concerned with the (lower) deferment price.

The argument just given relies on the idea that the ERM writer can implement dynamic hedging of the put it has written (or purchase the put from some other party who does so). In practice, this is not possible for a put on the price of a single house; the put cannot be hedged or traded. However, this objection is not specific to the argument just given; it is instead an objection to the whole paradigm of risk-neutral valuation of NNEG. But if we accept this paradigm, and also accept the assumption of a lower reflecting barrier, then the argument just given suggests we can set aside the second leg of Principle II.

For a crisp illustration of the argument, consider an 'ERM-let' where the rolled-up loan X_T (i.e. the strike price on the NNEG) is *below* the reflecting barrier. In this case, the price can never fall below the strike, so the NNEG can never have any payoff, and is clearly worth nothing; the value of the T-period 'ERM-let' is then just the value $e^{-rT}X_T$ of a loan with no NNEG. The price $e^{-rT}F_{0,T}$ of the prepaid forward (deferment price), however, is unaffected by the barrier, for the reasons explained in Section 4.1 of the paper. For a sufficiently long term., this deferment price will be lower than the value $e^{-rT}X_T$ the ERM writer is certain to receive. In this scenario, the deferment price is *not* a sensible upper bound on the value of the ERM.

A generalised form of Principle II to encompass the barrier case might be stated as:

> "The economic value of ERM cash flows cannot be greater than either the value of an equivalent loan without an NNEG, *or any related limit derived from put-call parity*."

[8] To see why this becomes small for long terms, note that the forward price allowing for the rental yield q on the house (which we have ignored in the main body of this paper) is $F_{0,T} = Se^{(r-q)T}$. Discounting this back to the present gives the present value of deferred possession as Se^{-qT}, which clearly reduces as T increases. (Other considerations besides rental yield might affect the present value of deferred possession of the property (e.g. illiquidity premiums, as discussed in IFoA Equity Release Mortgages Working Party, 2020), but I leave these aside in this paper.)